Anyon Superconductivity and the Fractional Quantum Hall Effect

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We obtain a hierarchy of effective Hamiltonians which allow for a unified treatment of the fractional quantum Hall effect and a gas of fractional-statistics particles (anyons) in two dimensions. Anyon superconductivity is the analog of the fractional quantum Hall effect. For a rational statistics parameter $\alpha_s = P/Q$ with PQ even, Q anyons bind forming a charge-Qe superfluid.

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Two-dimensional (2D) particles with fractional statistics, called anyons, were first introduced and studied by Wilczek in 1982.[1,2] Under exchange the anyon wave function acquires a phase factor $e^{i\alpha_s\pi}$ with the statistics parameter α_s noninteger. A physical realization of anyons was provided by the quasiparticles in the fractional quantum Hall effect.[3] More recently, Laughlin has argued that anyons may be playing a role in high-temperature superconductivity.[4] Indeed, a particular mean-field treatment predicts that a gas of noninteracting semions ($\alpha_s = \frac{1}{2}$) will pair up and be superconducting. This result is also supported by recent numerical calculations on small systems.[5]

In this Letter, we generalize a duality transformation, applied previously to 2D bosons,[6] to study the properties of an anyon gas. Specifically, starting with a lattice anyon Hamiltonian, a hierarchy of effective Hamiltonians with the same low-energy long-wavelength physics is obtained by successive duality transformations. As a check on this approach, we consider first the fractional quantum Hall effect (FQHE). The FQHE hierarchy[3,7] emerges in one-to-one correspondence with the hierarchy of Hamiltonians. Properties of the FQHE state can be calculated straightforwardly from the corresponding effective Hamiltonian. We find an incompressible liquid, quantized Hall conductivity, fractional-statistics quasiparticles, off-diagonal long-ranged order (ODLRO),[8] etc.

For the anyon gas an analogous hierarchy of possible states is likewise obtained, with statistics parameter expressed in the continued-fraction form

$$\alpha_s = \cfrac{1}{p_1 + \cfrac{1}{p_2 + \cdots + \cfrac{1}{p_n}}} \equiv \frac{P}{Q}, \tag{1}$$

with $p_i = 0, \pm 2, \pm 4, \ldots$, and $i = 1, \ldots, n$. All rationals with PQ even can be written in this way. Calculating with the effective Hamiltonian we find a $(T=0)$ state with the following properties: (i) A nonzero superfluid density, associated infinite longitudinal conductivity, σ_{xx}, and corresponding collective (massless) sound mode. (ii) A massive mode signifying a bound state of Q anyons and corresponding flux quantization with flux h/Qe. (iii) Vortices in the superconducting order parameter which interact logarithmically. (iv) A nonzero Hall conductivity σ_{xy}. These results are consistent with recent works by Fetter, Hanna, and Laughlin and Wen and Zee.[9] In contrast to the ODLRO in the FQHE state, anyon superconductivity survives for nonzero temperature, $T \neq 0$.

An anyon Wigner crystal state is also possible. For given interanyon interaction strength, our approach cannot ascertain the relative stability of the Wigner crystal and superconducting phase, but guided by the FQHE we expect superconductivity to be favored with short-ranged interactions and for lower levels in the hierarchy.[3,7]

First consider a rotor Hamiltonian describing bosons hopping on a 2D square lattice ($h = c = e = 1$): $H_1 = H_{1,t} + H_{1,u}$ with

$$H_{1,t} = -t_1 \sum_{r,\alpha} \cos(\Delta_\alpha \phi_{1,r} - 2\pi A^\alpha_{0,r}), \tag{2a}$$

$$H_{1,u} = \tfrac{1}{2} \sum_{r,r'} (N_{1,r} - \rho_1) u_1(r - r')(N_{1,r'} - \rho_1), \tag{2b}$$

where N_1, the Bose number operator, is conjugate to the phase ϕ_1; $[\phi_1, N_1] = i$. Here $u_1(r)$ is a repulsive interaction between bosons [e.g., $u_1(r) \sim e^2/r$] and the compensating positive-charge background per site, ρ_1, is taken much less than 1. The vector potential $\mathbf{A}_0$, corresponding to an applied magnetic field, couples to the lattice derivative, $\Delta_\alpha \phi_{1,r} \equiv \phi_{1,r+\alpha} - \phi_{1,r}$ with $\alpha = (x,y)$.

In order to describe anyons (or fermions) flux tubes are attached to each boson.[1,10] In (2a) we put $\mathbf{A}_0 \to \mathbf{A}_0 + \mathbf{a}_1$, with the statistical gauge field satisfying the constraint

$$(\nabla \times \mathbf{a}_1)_R = \frac{\alpha_s}{4} \sum_{r \in R} N_{1,r}, \tag{3}$$

and $\nabla \cdot \mathbf{a}_1 = 0$. Here $(\nabla \times \mathbf{a}_1)_R$ denotes a directed sum around the plaquette whose center is a dual-lattice site R, and $r \in R$ denotes the four nearest-neighbor real lattice sites to R. The statistics parameter α_s is an even integer for bosons, an odd integer for fermions, and noninteger for anyons.

In Ref. 6, we showed that the boson Hamiltonian H_1, when $\mathbf{A}_0=\mathbf{a}_1=0$, was isomorphic to a dual model describing (2+1)-dimensional (noncompact) scalar quantum electrodynamics (QED). The mapping to QED required a softening of the integer constraint on the eigenvalues of N_1. Consequently, the dual model contains some effective parameters which cannot be related quantitatively to those in (2). Nevertheless, the system's *qualitative* features, such as the long-wavelength, low-energy structure of the phases, should be described correctly by the effective dual Hamiltonian.

Under duality the original boson number operator $N_1 \rightarrow (\nabla\times\mathbf{A}_1)$, where $\mathbf{A}_1$, the QED gauge field, lives on the links of the dual lattice. The QED matter field, which lives on the sites of the dual lattice, is represented by a new boson number operator N_2 and associated conjugate phase ϕ_2. Generalizing the duality to include both $\mathbf{A}_0$ and $\mathbf{a}_1$ results in an effective Hamiltonian $H_2=H_{2,s}+H_{2,t}+H_{2,u}$ with

$$H_{2,s}=\frac{u_2}{2}\sum_R |\Pi_{1,R}|^2+H_{1,u} \quad (N_1=\nabla\times\mathbf{A}_1)\,, \tag{4a}$$

where Π_1 is the momentum conjugate to $\mathbf{A}_1$ ($\nabla\cdot\mathbf{A}_1=0$ is assumed),

$$H_{2,t}=-t_2\sum_{R,\alpha}\cos(\Delta_\alpha\phi_{2,R}-2\pi A^\alpha_{1,R})\,, \tag{4b}$$

$$H_{2,u}=\frac{u_2}{2}\sum_{R,R'}(N_{2,R}-\rho_{2,R})G(R-R')(N_{2,R'}-\rho_{2,R'})\,. \tag{4c}$$

In (4c) the 2D lattice Green's function $G(R)\sim -2\pi\ln R$ for large R and the operator

$$\rho_{2,R}=(\nabla\times\mathbf{A}_0)_R-\frac{\alpha_s}{4}\sum_{r\in R}(\nabla\times\mathbf{A}_1)_r\,. \tag{4d}$$

The operator N_2 in (4c) represents a vortex in the original boson field. As in the *classical* Coulomb-gas description of a boson film[11] these vortices interact logarithmically with one another. Since N_2 does not commute with ϕ_2, though, (4) represents a quantum field theory for this "charged" vortex plasma. The neutralizing background, which determines the number of vortices, is set by the effective magnetic field felt by the bosons, ρ_2 in (4d), which is a sum of the physical and statistical magnetic fields penetrating a lattice plaquette. The collective longitudinal sound mode of the boson system H_1 is described by (4a) (the photon in QED). The vortex hopping term (4b) couples together the bosons and vortices, and tends to make $\nabla\times\mathbf{A}_1$ quantized in integer units: This correctly accounts for the *discreteness* of the boson number N_1.

Consider the first hierarchy FQHE for fermions (with α_s an odd integer). When the density of electrons $\langle N_1\rangle=\langle\nabla\times\mathbf{A}_1\rangle$ is a (filling) fraction $\nu=1/\alpha_s$ of the applied magnetic field B_0, the effective magnetic field (4d) felt by the N_1 boson vanishes on average. This implies, in turn, $\langle N_2\rangle=0$. At this special filling, if the vortex hopping t_2 is small compared to u_2, one expects that positive ($N_2=+1$) and negative ($N_2=-1$) vortices should bind forming a "vortex insulator" with a gap. This corresponds to the FQHE state[12] (see below). In this representation, the order parameter for the FQHE[8] [$\langle\exp(i\phi_1)\rangle$ in (2)] becomes a "disorder" parameter for the vortex insulator.

The existence and properties of the FQHE state can be demonstrated explicitly by setting the hopping $t_2=0$, solving for the properties of the resulting quadratic Hamiltonian, demonstrating a gap for $N_2=\pm1$, and arguing that this gap survives small nonzero hopping. For $t_2=0$ eigenstates of H_2 can be expressed as a product $|\{N_2\}\rangle\otimes|\Psi_A\rangle$, yielding for a given set of integers $\{N_2\}$ an effective Hamiltonian $\tilde{H}_2(\{N_2\})$, which is *quadratic* in $\mathbf{A}_1$. With $\{N_{2,R}\}=0$ the resulting spectrum is massive:

$$\omega(k)=[(2\pi\alpha_s u_2)^2+u_1(k)u_2k^2]^{1/2}\,,$$

corresponding to incompressible density fluctuations [$\delta(\nabla\times\mathbf{A}_1)$].

Static Laughlin quasiparticles can be formed by choosing a nonzero set of $\{N_2\}$, with $+1$ (-1) corresponding to a quasiparticle (quasihole), and letting the fermion density $\nabla\times\mathbf{A}_1$ distort to find the ground state of $\tilde{H}_2(\{N_2\})$. With the $\{N_2\}$ constrained in this way one finds a ground-state energy

$$E(\{N_2\})=\tfrac{1}{2}\sum_{R,R'}N_{2,R}V(R-R')N_{2,R'}\,, \tag{5}$$

with a repulsive interaction $V(k)=(1/\alpha_s^2)u_1(k)$ for small wave vector k. Since $V(R=0)>0$, there is indeed a quasiparticle gap, thereby justifying our setting $t_2=0$ in the FQHE state. At large separations the quasiparticles interact with the same functional form as did the original fermions in (2b), but with a fractional charge $1/\alpha_s$. The origin of this fractional charge can be inferred directly from (4c) and (4d): Since $G(k)\sim 1/k^2$, the combination $N_2-\rho_2$ must vanish for small k, so that a site R with $N_{2,R}=1$ will necessarily be dressed by a cloud of fermion density ($\nabla\times\mathbf{A}_1$) with integrated charge $1/\alpha_s$. This charge cloud implies in turn fractional statistics[3] for the dressed quasiparticle. Indeed, when two static, but dressed, quasiparticles are interchanged adiabatically using the vortex hopping term (4b) they "see" each others fractional charge dressing as a gauge field A_1 corresponding to $1/\alpha_s$ of a magnetic flux quantum. Since a bare N_2 is a boson this implies $1/\alpha_s$ statistics.

The conductivity tensor $\sigma_{\alpha\beta}$ at $T=0$ follows by evaluating the correlation function

$$\sigma_{x\alpha}(i\omega)=\langle A_1^y(\mathbf{k}=0,\omega)J_1^\alpha(\mathbf{k}=0,-\omega)\rangle \tag{6}$$

in the ground state of the quadratic Hamiltonian $\tilde{H}_2(\{N_2=0\})$. Here τ (ω) denotes imaginary time (frequency) and $J_1^\alpha\equiv -i\epsilon_{\alpha\beta}\partial_\tau(\Delta_\beta\phi_2-2\pi A_1^\beta)$ is the fermion current operator, obtained from the duality mapping by differentiating the imaginary-time action with respect to A_0^s. Upon expressing $\partial_\tau\phi_2$ in terms of $\mathbf{A}_1$ and N_2 using $\partial_\tau\phi_2=[\phi_2,H_2]$, a simple Gaussian integral (over $\mathbf{A}_1$) yields the expected result $\sigma_{\alpha\beta}=(1/\alpha_s)\epsilon_{\alpha\beta}$.

A simple physical understanding of σ_{xy} quantization follows readily from the Hamiltonian H_2. It is apparent from (4c) and (4d) that the statistical flux tubes will tend to induce α_s (an odd integer) negative vortices *near* each fermion whereas the applied magnetic field creates an equal number of positive vortices. In the FQHE state these positive vortices bind to the negative ones (creating a vortex insulator), thus effectively binding to the fermions. More generally, for filling factors $v\equiv P/Q$ satisfying the hierarchy condition, one can show that P fermions bind with Q vortices. The binding of vortices to fermions in the FQHE state was pointed out a number of years ago by Halperin.[13] As noted recently,[8] the bound composite object effectively undergoes Bose condensation. The associated order parameter, $\langle\exp(i\phi_1)\rangle$ in (2), can indeed be shown to be nonzero in the FQHE state. Within this picture, quantization of σ_{xy} is a direct consequence of the Josephson relation: For given fermion current I, the resulting vortex current I/v induces 2π phase slips (in ϕ_1) causing a transverse voltage $V_H=I/v$.

To obtain the FQHE hierarchy we iterate the above mapping. First, tie flux tubes to the vortices N_2 in (4) with an even integer, p_1, of flux quanta. This preserves the bosonic character of N_2, allowing a second duality mapping to be performed. An effective Hamiltonian is thereby obtained with the form

$$H_3=H_{2,s}+H_{3,s}+H_{3,t}+H_{3,u}\,. \tag{7}$$

Here the last three contributions are identical in form to (4a)-(4c), but with all subscripts increased by one, $r\leftrightarrow R$, and the statistics angle α_s replaced by an even integer p_1. Note that the vortex number operator is $N_2\to(\nabla\times\mathbf{A}_2)$ under this second mapping. This procedure can be further iterated obtaining, for general n, a Hamiltonian $H_n=H_{2,s}+\cdots+H_{n,s}+H_{n,t}+H_{n,u}$, which is a function of operators ϕ_n, N_n, and $A_1,\ldots,A_{n-1}$. For n odd (even) matter fields $e^{i\phi_n}$ live on the sites of the r (R) lattice, whereas gauge fields A_n live on the links of the R (r) lattice.

The allowed filling fraction v for the nth level of the FQHE hierarchy can be readily obtained by inspection of H_{n+1}. The ratio of the magnetic field strength $B_0=\nabla\times\mathbf{A}_0$ to the fermion density $\rho_1=\langle\nabla\times\mathbf{A}_1\rangle$ must be chosen to obtain a zero neutralizing background density for the N_{n+1} "particles," enabling a paired insulator to form. For example, for the second hierarchy this condition implies that $\rho_1=p_1\langle\nabla\times\mathbf{A}_2\rangle$ and $\langle\nabla\times\mathbf{A}_2\rangle=B_0-\alpha_s\rho_1$. Eliminating $\langle\nabla\times\mathbf{A}_2\rangle$ yields $v^{-1}=\alpha_s+1/p_1$. Properties of the associated FQHE state can be obtained from H_{n+1} by putting $\{N_{n+1}\}=0$ and turning off the hopping t_{n+1}. The resulting Hamiltonian is quadratic (in $\mathbf{A}_1,\ldots,\mathbf{A}_n$) and relevant correlation functions can be readily computed.

To apply the hierarchy of effective Hamiltonians to study the anyon gas, we set the external magnetic field to zero and consider noninteger α_s. With $\nabla\times\mathbf{A}_0=0$, the neutralizing background $\langle\rho_2\rangle$ in H_2 in (4d) does *not* vanish for any α_s. However, in the effective Hamiltonian H_3 in (7), the neutralizing background, $\langle\rho_3\rangle=\langle\nabla\times\mathbf{A}_1\rangle-p_1\langle\nabla\times\mathbf{A}_2\rangle$, for the N_3 particle vanishes provided that $\langle\nabla\times\mathbf{A}_1\rangle=p_1\langle\nabla\times\mathbf{A}_2\rangle$, with p_1 an even integer. Since the $H_{3,s}$ term in (7) requires $\langle\nabla\times\mathbf{A}_2\rangle=\alpha_s\langle\nabla\times\mathbf{A}_1\rangle$, this condition is satisfied for $\alpha_s=1/p_1$.

Thus for statistics parameter $\alpha_s=1/p_1$ the neutralizing background for the N_3 particle vanishes and, as in the second hierarchy FQHE, a paired insulator of the N_3 bosons should result (for small t_3). The properties of the resulting anyon state can be deduced by setting the N_3 hopping, t_3, to zero, precisely as in the FQHE.

With $t_3=0$, the eigenstates can be expressed as $|\{N_3\}\rangle\otimes|\Psi_{A_1,A_2}\rangle$, so that for a given set of $\{N_3\}$ an effective Hamiltonian $\tilde{H}_2(\{N_3\})$, quadratic in the gauge fields $\mathbf{A}_1,\mathbf{A}_2$ is obtained. The ground-state energy $E(\{N_3\})$ of $\tilde{H}_3(\{N_3\})$ gives the effective interaction between the N_3 particles. We find

$$E(\{N_3\})=\tfrac{1}{2}\sum_{r,r'}(\alpha_s^2u_2u_3/u_{23})G(r-r')N_{3,r}N_{3,r'}\,, \tag{8}$$

where $u_{23}=\alpha_s^2u_2+u_3$. Since $G(r)$ is repulsive, configurations with nonzero $\{N_3\}$ are separated by an energy gap from the N_3 vacuum: The phase with $t_3\equiv 0$ will thus survive small nonzero hopping. Moreover, properties of the phase can be deduced with $t_3=0$.

When $\{N_{3,r}\}=0$, the eigenspectra of $\tilde{H}_3(\{N_3\}=0)$ consist of a gapless longitudinal sound mode $\mathbf{A}_{1,q}$ with long-wavelength dispersion,

$$\omega_{1,q}=[u_1(q)u_2u_3/u_{23}]^{1/2}|\mathbf{q}|\,,$$

and a massive mode with frequency $\omega_{2,q}=2\pi u_{23}/|\alpha_s|$. For short-range repulsive interaction between the anyons, the sound mode is linear, $\omega_1(q)\sim|\mathbf{q}|$. This is suggestive of superfluidity.

Superfluidity of the anyon-gas ground state can be confirmed directly by calculating $\sigma_{\alpha\beta}$. Evaluating (6) in the ground state of $\tilde{H}_3(\{N_3\})$ (noting that $N_2\to\nabla\times\mathbf{A}_2$) yields

$$\sigma_{xx}(\omega)=\frac{2\pi}{i\omega}\frac{u_2u_3}{u_{23}}\,. \tag{9}$$

The $1/\omega$ pole indicates a nonzero superfluid density. In contrast to a conventional superfluid, though, we find a nonzero $\sigma_{xy}(\omega=0)=\alpha_s^3u_2^2/u_{23}^2$, which is allowed since the anyon gas is not time-reversal invariant. Because of the $1/\omega$ pole in (9), though, the Hall resistivity ρ_{xy} *vanishes* at $\omega=0$.

We can deduce the number of anyons in the composite boson responsible for superfluidity in (9) by examining the flux quantization in an applied magnetic field. First, we identify the logarithmically interacting N_3 particles with vortices in the anyon superconducting wave function. In an external magnetic field, $\nabla\times\mathbf{A}_0$, "charge neutrality" in H_3 implies $\langle N_3\rangle = p_1\langle\nabla\times\mathbf{A}_0\rangle$, i.e., a magnetic flux of hc/p_1e per N_3 vortex. Flux quantization with hc/p_1e implies, in turn, that p_1 anyons have bound and Bose condensed. (The massive mode, with frequency $\omega_{2,q}$, is a direct indication of this.) For semions the condensate boson is made from two anyons, as predicted by Laughlin.[4]

For anyons with statistics parameter $\alpha_s \neq 1/p_1$, it is necessary to go further up the hierarchy to find a vanishing neutralizing background density. By doing so one can construct a hierarchy of anyon superconducting states, analogous to the FQHE hierarchy. We find that for α_s ($\equiv P/Q$) satisfying the continued-fraction expansion in (1), a charge-Qe superfluid state can form, with Q anyons binding to form the condensate boson and flux quantization with flux hc/Qe. The hierarchy of α_s in (1) is the same as the hierarchy for the filling ν in the boson FQHE. Note that both even *and* odd Q are allowed by (1), although PQ must be even.

Since the vortices in the anyon superconducting wave function interact logarithmically [see (8)], anyon superconductivity should survive at $T\neq 0$. This should be contrasted with the FQHE state, which loses ODLRO (and power-law order) at *any* $T\neq 0$ due to unbinding of the (N_2) vortices in (5), which have a finite (not infinite) pair-breaking energy. As T is raised the anyon superconductor will presumably undergo a Kosterlitz-Thouless vortex-unbinding transition[11] into a normal phase.

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