Gauge Theory of Two-Dimensional Quantum Gravity

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We present a gauge theory of two-dimensional gravity which is derived from a generalization of the Jackiw-Teitelboim model. Using canonical quantization, we construct the exact quantum solution of this model.

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Lower-dimensional theories of gravity have attracted a growing interest recently. While such theories are mostly studied to gain insight into both the conceptual and technical issues that arise in the quantization of gravity in four dimensions, they also possess physical and mathematical significance. In two dimensions the usual Einstein-Hilbert action is a topological invariant (the Euler class) of the space-time manifold; the Einstein tensor vanishes identically. As a natural analog of the vacuum Einstein equations, Jackiw and Teitelboim[1] proposed the equation $R-2k=0$, where R is the curvature scalar and k is the cosmological constant. Moreover they proposed an action which yields this equation upon a local variational principle. Soon after, an exact solution of this model for the case of open spatial sections was found.[2] Quantization of the (topological) Einstein-Hilbert action[3] and of the action induced by massless particles[4] have also been studied.

In three dimensions the Einstein-Hilbert action is no more a topological invariant. However, the theory is still "trivial" in the sense that there are no propagating degrees of freedom. It has been shown[5] that three-dimensional gravity can be formulated as a Chern-Simons gauge theory of ISO(2,1), making the exact solution of the model possible.

It is the purpose of this Letter to combine the above group theoretical ideas about three-dimensional gravity with the Jackiw-Teitelboim model of two-dimensional gravity. We present a generalization of the Jackiw-Teitelboim model which can be formulated, for nonvanishing cosmological constant k, as a gauge theory of SO(2,1). We perform the canonical quantization and construct the exact quantum solution of this gauge theory of two-dimensional gravity.

We formulate our model in terms of the *Zweibein* and spin-connection one-forms e^a $(a=0,1)$ and ω, taken as independent variables and combined into a gauge-field one-form of the Poincaré group ISO(1,1),

$$A=e^aP_a+\omega\Lambda\,. \tag{1}$$

P_a are the generators of translations and Λ is the generator of Lorentz transformations. Together they satisfy the two-dimensional Poincaré algebra

$$[\Lambda,P_a]=\epsilon_a{}^bP_b,\quad [P_a,P_b]=0\,. \tag{2}$$

Our conventions for the flat Minkowski metric and the antisymmetric ϵ tensor are $\eta_{ab}=\mathrm{diag}(-1,1)$, $\epsilon_{01}=1$. The generator of Lorentz transformations acting on two-vectors is $-\epsilon^a{}_b$. The curvature two-form constructed from the connection (1) is

$$F=dA+A\wedge A=\mathcal{T}^aP_a+\mathcal{R}\Lambda\,, \tag{3}$$

where $\mathcal{T}^a=de^a-\omega\epsilon^a{}_b\wedge e^b$ and $\mathcal{R}=d\omega$ are the torsion and curvature two-forms, respectively.

It is not possible to formulate an ISO(1,1) gauge theory, since there is no invariant, nondegenerate bilinear form on the Poincaré algebra (except in three dimensions[5]). In the presence of a nonvanishing cosmological constant k, we can, however, use this dimensionful parameter to deform the Poincaré algebra to the de Sitter algebra

$$[\Lambda,P_a]=\epsilon_a{}^bP_b,\quad [P_a,P_b]=k\epsilon_{ab}\Lambda\,. \tag{4}$$

This algebra possesses an invariant, nondegenerate bilinear form given by the Killing metric

$$g_{ij}=\begin{pmatrix} k\eta_{ab} & 0\\ 0 & 1\end{pmatrix},$$

where the new indices i,j run from 0 to 2. Using this metric and the definition $(T_i)=(T_0,T_1,T_2)=(P_0,P_1,\Lambda)$ we can write (4) in the compact form

$$[T_i,T_j]=f_{ij}{}^kT_k=\epsilon_{ijk}g^{kl}T_l\,,$$
$$g_{ij}=\tfrac{1}{2}f_{ik}{}^lf_{jl}{}^k,\quad \epsilon_{012}=1\,. \tag{5}$$

The Killing metric becomes degenerate in the limit $k\to 0$. For Minkowskian space-time, (5) is the SO(2,1) (de Sitter) algebra, whereas for Euclidean space-time and $k>0$ it becomes the SO(3) algebra.

The two-component of the curvature, corresponding to the original Lorentz generator Λ, becomes now

$$F^{i=2}=d\omega+\frac{k}{2}\epsilon_{ab}e^a\wedge e^b. \tag{6}$$

The condition of vanishing curvature, $F=0$, is therefore equivalent to the equations of motion of the Jackiw-Teitelboim model[1] (expressed in terms of the *Zweibein* and the spin connection),

$$F^i=0 \Longleftrightarrow \begin{cases} T^a=0, \\ R=2k, \end{cases} \tag{7}$$

where $R=(2/\det e)\epsilon^{\mu\nu}\partial_\mu\omega_\nu$ is the Ricci curvature scalar. Using the trace on the Lie algebra given by the Killing metric, i.e., $\mathrm{Tr}T_iT_j=g_{ij}$, we can produce (7) as the equations of motion for the action

$$S=\int \mathrm{Tr}\varphi F=\int d^2x\,\mathrm{Tr}\varphi(\partial_0A_1-\partial_1A_0+[A_0,A_1]). \tag{8}$$

φ is a dimensionless auxiliary field with values in the Lie algebra, which satisfies the equation of motion $D\varphi=0$.

The action is invariant with respect to infinitesimal gauge transformations with parameter $\lambda=\lambda^iT_i$,

$$\begin{aligned} \delta A&=-d\lambda+[\lambda,A]=-D\lambda, \\ \delta\varphi&=[\lambda,\varphi]. \end{aligned} \tag{9}$$

Under a diffeomorphism $\delta x^\mu=\epsilon^\mu(x)$, the gauge field transforms with the Lie derivative, which can be written[6] as

$$\delta A_\mu=\epsilon^\rho F_{\rho\mu}+D_\mu(\epsilon^\rho A_\rho). \tag{10}$$

Using the equations of motion $F=0$, this is equivalent to a gauge transformation with parameter $\lambda=-\epsilon^\rho A_\rho$. The same is true for the auxiliary field φ. Thus we see that on-shell, gauge invariance is equivalent to general coordinate invariance (see also Ref. 5).

We now proceed to the canonical quantization of the theory specified by the action (8) on a space-time with closed circles of length L as spatial sections. The quantization of the first-order Langrangian corresponding to (8) (in phase-space form) leads to nontrivial equal-time commutation relations between φ and A_i,[7]

$$[\varphi_i(t,x),A_1{}^j(t,y)]=\frac{1}{i}\delta(x-y)\delta_i{}^j. \tag{11}$$

The nondynamical Lagrange multiplier A_0 leads to the "Gauss-law" constraint

$$D_1\varphi=\partial_1\varphi+[A_1,\varphi]=0. \tag{12}$$

As a consequence of the canonical algebra (11), Gauss's law generates gauge transformations on the dynamical field A_1. As always in generally covariant theories the Hamiltonian is proportional to the constraints and vanishes weakly (as indicated by $\approx$)

$$H=-\int_0^L dx\,\mathrm{Tr}A_0D_1\varphi\approx 0. \tag{13}$$

Note that spatial partial integrations do not produce any surface term, since we assume closed spatial sections. According to the canonical algebra (11) the Hamiltonian (13) generates the equations of motion for the dynamical field A_1, $\partial_tA_1=i[H,A_1]=D_1A_0$. Thus we see that the effect of introducing the field φ and the action (8) is to eliminate the original coordinate A_0 (which now becomes a nondynamical Lagrange multiplier) in favor of a canonical momentum φ conjugate to the dynamical variable A_1.

In order to give a functional Schrödinger representation[8] for the canonical algebra (11) we choose the "position-space polarization" in which states are represented by functionals of the coordinate A_1, $\Psi(A_1)$. Then A_1 acts on the states by multiplication and φ by functional differentiation, $\varphi_i\Psi(A_1)=(\delta/i\delta A_1^i)\Psi(A_1)$. Gauge invariance at the quantum level is guaranteed by the requirement that the Gauss-law constraint (generator of gauge transformations on A_1) annihilates physical states. This functional differential equation is most easily solved by the introduction of the Lie-group-valued function $S(x)$ (see also Ref. 9),

$$\begin{aligned} A_1(x)&=S(x)\partial_xS^{-1}(x), \\ S(x)&=P\exp\left[-\int_0^x dx'A_1(x')\right], \end{aligned} \tag{14}$$

where we have omitted the time variable and P denotes the usual path ordering. $W\equiv S(L)$ is then the Wilson-loop variable around the closed spatial sections. Because of

$$\left[\partial_x\frac{\delta}{\delta A_1^i(x)}+\epsilon_{ij}{}^lA_1^j(x)\frac{\delta}{\delta A_1^l(x)}\right]\mathrm{tr}W=0, \tag{15}$$

the gauge-invariant physical states are given by *functions* $\psi(\mathrm{tr}W)$, where the trace is taken over an irreducible representation of the gauge group. Note that in the case of infinite-dimensional representations of SO(2,1) the trace converges in the distribution sense.[10] The solution (15) of the Gauss-law constraint follows also from gauge invariance of the Lagrangian (specifically there is no total time derivative term in the transformation of the Lagrangian). According to a general result,[11] in this case there is no one-cocycle in the realization of gauge invariance on the physical states. Therefore, these must satisfy $\Psi(A_1^g)=\Psi(A_1)$ and the only gauge-invariant functionals of A_1 are built from the trace of the Wilson loop around the closed space.

The above computation shows that the configuration space of two-dimensional quantum gravity is given by the gauge group G modulo its adjoint action, G/G_{adj}, where $G=\mathrm{SO}(3)$, SO(2,1) for $k>0$ and Euclidean and Minkowski gravity, respectively. In the case of Euclidean gravity, with the compact gauge group SO(3), the Peter-Weyl theorem[12] states that a complete basis for the square integrable functions on the above configuration space is given by the character functions $\chi_r(W) \equiv \mathrm{tr}_r(W)$ of the irreducible representation r of SO(3):

$$\psi = \sum_r \lambda_r \chi_r \,. \tag{16}$$

For Minkowski gravity, with the noncompact gauge group SO(2,1), an analogous construction exists.[10] However, in this case one must take into account also the infinite-dimensional representations of SO(2,1).

Since the principle of our construction is to substitute general coordinate invariance with gauge invariance, we can produce dynamics for our theory by adding to the action a gauge-invariant potential term for φ, $-\int d^2x \times V(\mathrm{tr}\varphi^2)$ (we recall that φ is the momentum conjugate to A_1). This produces the Hamiltonian $H = \int_0^L dx \times V(\mathrm{tr}\varphi^2)$ and changes the equations of motion to

$$F^i_{01} = V'(\mathrm{tr}\varphi^2) 2\varphi^i \,, \tag{17}$$

where the prime denotes differentiation with respect to the argument. Note that at all minima of the potential V this equation of motion requires the vanishing of the curvature and reproduces the equations of the Jackiw-Teitelboim model. Using symmetric point splitting we find

$$\frac{\delta}{\delta A^i_1(x)} \frac{\delta}{\delta A_{1i}(x)} \mathrm{tr}_r W = C_r \,\mathrm{tr}_r W \,, \tag{18}$$

where C_r is the quadratic Casimir of the representation r. Therefore, the character functions χ_r are eigenstates of the Hamiltonian with eigenvalue given by

$$H\chi_r = LV(C_r)\chi_r \,. \tag{19}$$

The ground state of the theory has vanishing torsion and constant curvature.

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