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Characterization of Unstable Periodic Orbits in Chaotic Attractors and Repellers

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A numerical technique for the characterization of the chaotic regime of dissipative maps through unstable periodic orbits is presented. It is shown that although the maps are dissipative their trajectories can be derived from a Hamiltonian, which allows us to calculate unstable periodic orbits *of arbitrary length* finding *all* points to any desired accuracy. Applying the method to the Hénon map we find that in a previously unexplored region of parameter space the topological entropy exhibits plateaus on which it is constant while the dynamics is characterized by a strange repeller.

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Chaotic behavior in dynamical systems has been a subject of extensive study in recent years. The transitions from regular to chaotic motion have been classified and their universal properties have been considered using numerical simulations and renormalization-group techniques.[1] More recently, the emphasis has shifted to the chaotic regime itself.[2-8] In dissipative systems this regime is characterized by strange attractors.[9] The trajectories on the attractor exhibit positive Lyapunov exponents, and therefore they are exponentially sensitive to the initial conditions.

Recently, it was suggested that strange attractors can be studied by considering the unstable periodic orbits associated with them.[2-8] Such orbits provide a hierarchical framework, based on their lengths, which is useful in calculations of dimensions and entropies. In order to perform such calculations one has to find the set of all periodic orbits up to as high an order as possible. A numerical algorithm for the calculation of periodic orbits of a given map was proposed,[2] which is based on the identification of approximate orbits from a set of trajectories of the map and subsequent improvement by Newton-Raphson iterations.

In this Letter we present a new algorithm, which allows the calculation or *arbitrarily long* periodic orbits to *any* desired accuracy, for a certain class of maps. It allows us to identify the periodic orbits of a given order and calculate any particular one. The method is applied to the Hénon map, for which we compute all periodic orbits up to order $p=28$ (for $a=1.4$, $b=0.3$), and selected orbits up to order $p=1000$. We demonstrate that the method is useful even in the previously inaccessible region $1.42<a<2.65$, $b=0.3$ (Ref. 10), where we show that the map exhibits a strange repeller. Strange repellers are useful in the study of transient chaotic behavior.[11] Using our technique we have computed the topological entropy and estimated fractal dimensions in both the attractor and the repeller regions.

Our method is based on the observation that the dynamics of maps such as the Hénon map, although dissipative, can be derived from a Hamiltonian. This Hamiltonian is constructed in such a way that its spatial behavior is equivalent to the temporal behavior of the map. In particular, it is shown that there is a one-to-one correspondence between the trajectories of the map and the extremum static configurations of the Hamiltonian. This equivalence applies in both the regular and the chaotic regimes. Therefore, by calculating the extremal configurations of the corresponding Hamiltonian, unstable trajectories of the map can be found to any desired accuracy even in the chaotic regime.

We now illustrate the method in the case of the Hénon map. We find that the Hamiltonian associated with this map (i.e., for which the static Euler-Lagrange equations reproduce the map) takes the form[12]

$$H=H_k+H_p\,, \tag{1}$$

where the kinetic term is

$$H_k=\tfrac{1}{2}\sum_n b^{-n}(dx_n/dt)^2 \tag{2}$$

and the potential is

$$H_p = \sum_n (-b)^{-n} [x_n(x_{n+1} - x_{n-1}) - (b^{-1}+1)(ax_n - \tfrac{1}{3}x_n^3)] . \quad (3)$$

This Hamiltonian can be interpreted as describing an infinite chain of atoms interacting with a potential and among themselves. Here, x_n is the position of the nth atom while $a>0$ and $0<b\le 1$ are parameters. In this Letter we are interested only in static extremum configurations of (1) and therefore the kinetic term H_k will be neglected.[13,14] The potential energy (3) contains two terms: The first one describes the interactions among the atoms while the second term describes the interaction with the underlying potential. Note that the nth atom interacts only with the $(n-1)$th and $(n+1)$th atoms, which are not necessarily its nearest neighbors in configuration space. In this Hamiltonian the ordinary cyclic permutation symmetry $x_n \to x_{n+1}$ is replaced by a lower symmetry which is a combination of permutation and a rescaling transformation of the form $x_n \to x_{n+1}$, $H_k \to b^{-1}H_k$, and $H_p \to (-b)^{-1}H_p$. This property reflects the dissipative nature of the corresponding map. The potential (3) is not bounded from below and, therefore, this model does not have a ground state but only metastable states. The force on the nth atom is given by

$$F_n = (-b)^{-n}(b^{-1}+1)[-x_{n+1}+a-x_n^2+bx_{n-1}] . \quad (4)$$

When the chain is in stable or unstable equilibrium [namely, an extremum static configuration of (1)], $F_n=0$ for all n. In this case (4) is the static Euler-Lagrange equation associated with (1). This set of equations is equivalent to the Hénon map in the sense that every trajectory of the Hénon map obeys (4) with $F_n=0$, and vice versa. This can be easily seen by taking the Hénon map, which takes the form

$$x_{n+1} = a - x_n^2 + by_n , \quad y_{n+1} = x_n , \quad (5)$$

and eliminating y_n for all n.

We now show how the model (1) can be used to perform exact calculations in the chaotic regime of the Hénon map. In particular, consider the calculation of periodic cycles. To find a specific extremum configuration of order p of the Hamiltonian we introduce an artificial dynamics defined by

$$dx_n/dt = s_n F_n, \quad n=1,\ldots,p , \quad (6)$$

where $s_n = \pm 1$.[15] Then we solve Eqs. (6) subject to the periodic boundary condition $x_{p+1}=x_1$. This drives the system towards the desired extremum associated with the given set of $\{s_n\}$. When the forces on all the atoms decrease to zero the resulting structure x_n, $n=1,\ldots,p$, is simultaneously an extremum static configuration of (1) and an exact periodic orbit of the Hénon map.

Our aim in this calculation is to find all the periodic configurations of order p. In these extremum configurations each atom can sit either at a local minimum or at a local maximum of the energy, allowing in principle for 2^p configurations of order p. In order to find a desired configuration one has to choose $s_n=1$ for all atoms which are at local minima and $s_n=-1$ for atoms which are at local maxima. We find that such configurations are unique (namely, there is no more than one extremum configuration with the same sequence of minima and maxima, up to cyclic permutations), and that the method described above converges to them. In general, given $\{s_n\}$, the corresponding periodic configuration does not always exist. In this case the atoms escape to infinity since the potential is not bounded from below. In short, our technique finds all the periodic orbits which exist and tells which ones do not.

Consider a periodic configuration $\{x_n\}$, associated with a specific choice of $\{s_n\}$. Let us define

$$S_n = (-1)^n s_n, \quad n=1,\ldots,p . \quad (7)$$

It turns out that for most trajectories the sequence S_n coincides with the symbolic dynamics[6] $\bar{S}_n$ of the Hénon map, which is defined by $\bar{S}_n=+1$ if $x_n>0$ and $\bar{S}_n=-1$ if $x_n<0$.[16] This connection between S_n and $\bar{S}_n$ has important implications since it establishes a relation between the structure of the dynamic trajectory and the energetics of the underlying Hamiltonian. Also, for the first time it provides a systematic method for calculations in the chaotic regime, in which one can examine any particular orbit identified by $\{S_n\}$. Map iteration techniques do not provide such a systematic framework. Moreover, such methods can be applied only for short unstable cycles since the numerical error grows exponentially with the length of the cycle.[17] Our method does not suffer from these limitations and it can be used for arbitrarily long cycles to any desired accuracy.

In practice we solve Eqs. (6) until either all forces become smaller than a test value ($|F_n|<\epsilon$, where typically $\epsilon=10^{-7}$), or the atoms escape to infinity. The procedure converges for all initial conditions as long as $|x_n|$, $n=1,\ldots,p$, are small with respect to $\sqrt{a}$. Since only the final configuration is of interest it is possible to choose a simple Runge-Kutta method with a relatively large step size ($h=0.1$) to solve the equations. A similar technique was previously applied to the study of critical behavior in the Frenkel-Kontorova model.[18] To test the procedure we computed selected periodic orbits of up to order $p=1000$. For $p=1000$ the Sun 3/60 computer needs 1.3 s to find a specific extremum configuration. Using our method we have obtained the following significant results for the Hénon map. For $a=1.4$, $b=0.3$ we calculated all periodic orbits up to order $p=28$ (Table I). The method allows us to eliminate the calculation of cyclic permutations as well as orbits which are repetitions of lower-order cycles, leading to savings in computation time of at least a factor p. We have calcu-

TABLE I. The number of unstable periodic orbits of the Hénon map for $a=1.4$, $b=0.3$ for orders $p>10$. $N_c(p)$ is the number of orbits of order p excluding cyclic permutations and repetitions of lower cycles, $N(p)$ is the total number of periodic points of order p and its divisors, and $K_0(p)$ is the pth-order approximation to the topological entropy.

p	$N_c(p)$	$N(p)$	$K_0(p)$
11	14	155	0.661446
12	19	247	0.662364
13	32	417	0.669531
14	44	647	0.666973
15	72	1081	0.671877
16	102	1695	0.670442
17	166	2823	0.674295
18	233	4263	0.669870
19	364	6917	0.671365
20	535	10807	0.669984
21	834	17543	0.671362
22	1225	27107	0.669381
23	1930	44391	0.671216
24	2902	69951	0.670586
25	4498	112451	0.671157
26	6806	177375	0.670632
27	10518	284041	0.670953
28	16031	449519	0.670644

lated the topological entropy, defined by[19]

$$K_0=\lim_{p\to\infty} p^{-1}\log_2[N(p)-1]\,, \tag{8}$$

where $N(p)$ is the number of points which belong to periodic orbits of order p and its divisors.[20] We find that the topological entropy is $K_0=0.6708\pm0.0003$ (see Table I). Our results up to order 12 are in agreement with those obtained earlier.[2]

Previous studies of the Hénon map have been focused on specific values of the parameters, like $a=1.4$ and $b=0.3$ where the map exhibits a strange attractor. It was pointed out[10] that in some other regions, such as $1.42<a<2.65$, $b=0.3$ the trajectories escaped to infinity for all choices of initial conditions and, therefore, no further studies have been done.

Since our method is not sensitive to dynamical instabilities it is also useful in the previously inaccessible region. Here we demonstrate that, in fact, the dynamics in this region is characterized by a strange repeller. We construct the periodic orbits in this region and calculate the topological entropy for orbits up to order $p=15$ (Fig. 1). We find that K_0 is a monotonically increasing function of a. Surprisingly there are regions (such as $1.65<a<2.0$) where K_0 is constant indicating that the structure of the periodic orbits is independent of a. The unstable periodic orbits associated with the strange repeller at $a=1.8$, $b=0.3$ are shown in Fig. 2. Note that the repeller has the same folded structure as the attractor but unlike the attractor it exhibits an apparent structure of gaps.

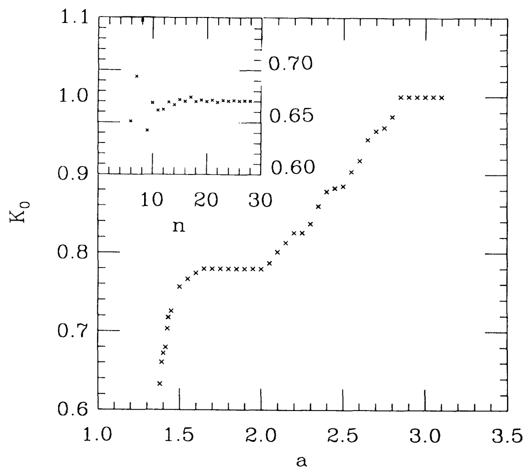

FIG. 1. The topological entropy of the Hénon map for $b=0.3$ as a function of a. Orbits of order up to 15 were included in the calculation. Note the existence of plateaus, i.e., for $1.6<a<2$, and the saturation of K_0 to its maximum value of 1 for $a>2.9$. Inset: The approximations to K_0 with increasing order p, for $5<p<29$ ($a=1.4$).

We have calculated the Grassberger-Procaccia (GP) dimensions for both the attractor $a=1.4$, $b=0.3$, and the repeller $a=1.6$, $b=0.3$ (Fig. 3), by using all periodic points of order 12, 15, and 18 and embedding the resulting sequence in a three-dimensional space. Calculating the correlation integral $C(r)$ (Ref. 21) we have estimated the GP dimensions as $\nu=1.24\pm0.33$ ($a=1.4$) and $\nu=1.22\pm0.04$ ($a=1.6$).

In summary, we have presented a useful method for calculations in the chaotic regime of dissipative maps. In this method one constructs a Hamiltonian such that the set of extrema of the Hamiltonian is equivalent to the set

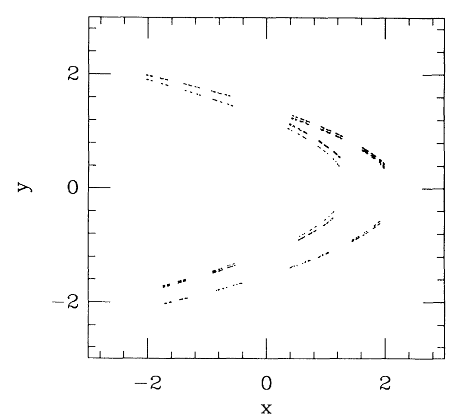

FIG. 2. A repeller for the Hénon map at $a=1.8$, $b=0.3$. Comparing this figure with a corresponding figure for the attractor, we note the existence of gaps in the set, which are similar in structure to preimages of the escaping region in the Feigenbaum map at $a>4$.

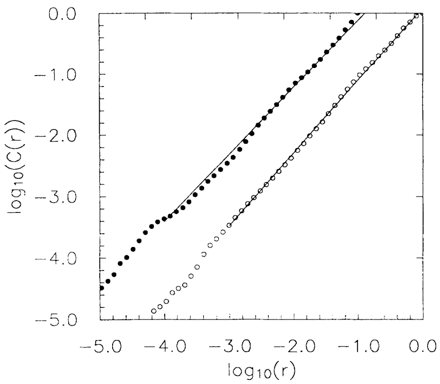

FIG. 3. The correlation integral $C(r)$ vs the distance r on a logarithmic scale (base 10). Open (closed) circles refer to parameter values of $a=1.4$ (1.6), $b=0.3$, respectively. We used periodic orbits of up to order 18 to construct 1834 (1232) vectors of embedding dimension 3. From these data we estimate the Grassberger-Procaccia dimensions to be $\nu=1.24\pm0.03$ ($a=1.4$) and $\nu=1.22\pm0.04$ ($a=1.6$).

of trajectories of the map. Using this method for the Hénon map we were able to calculate unstable periodic cycles of arbitrary length to any desired accuracy. We have shown that in a large and previously unexplored region of the parameter space, the dynamics is characterized by a strange repeller. Calculating the topological entropy K_0 in the repeller region we found that it exhibits plateaus where it is independent of the parameters of the map.

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[12]Note that we do not present a systematic method for the construction of the Hamiltonian. In fact, the Hamiltonian was first found by inspection. A more systematic approach will be presented in O. Bilham, C. Jayaprakash, and W. Wenzel (to be published) in which results for other maps, such as the dissipative standard map, will be reported.

[13]It turns out that the dynamics of such models is also of interest. The dynamics and its relevance for the corresponding maps will be discussed in Bilham, Jayaprakash, and Wenzel (Ref. 12).

[14]Biham, Jayaprakash, and Wenzel, Ref. 12.

[15]In fact, in the numerical calculations one can neglect the prefactors in Eq. (4) replacing F_n by

$$\bar{F}_n=(-1)^n[-x_{n+1}+a-x_n^2+bx_{n-1}]\,.$$

[16]It was pointed out that in some cases the symbolic dynamics $\bar{S}_n$ is not unique; namely two different cycles may have the same symbolic dynamics (Ref. 6). We find that even in such cases the sequences S_n given by (7) are unique and can be used to identify the cycles. Therefore, we propose that these sequences can be used as an alternative definition of the symbolic dynamics.

[17]Suppose that the computer precision is $\delta=10^{-k}$. Since the errors after n iterations increase exponentially like λ^n where $\lambda>1$, any roundoff error expands to order unity after $n=\log_\lambda(1/\delta)$ iterations. This sets an upper bound on the length of unstable periodic cycles which can be calculated.

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[19]We find it most convenient to take a logarithm to the base 2, instead of e. With this choice we obtain that inside the period-doubling cascade $K_0=0$ while for $a\to\infty$, where $N(p)=2^p$, $K_0=1$.

[20]Note that the number of periodic points which belong to cycles of order p and its divisors is $N(p)=\sum iN_c(i)$, where $N_c(i)$ is the number of periodic cycles of order i and the sum is over all the divisors of p including 1 and p.

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