Coherent-State Path Integrals and the Bosonization of Chiral Fermions

Michael Stone
Department of Theoretical Physics, Schuster Laboratory, University of Manchester, Manchester M13 9PL, England and Loomis Laboratory of Physics, University of Illinois, 1110 West Green Street, Urbana, Illinois 61801 [(a)]
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By using coherent-state path integrals I provide a derivation of a two-dimensional chiral boson action which is equivalent to a free Weyl fermion.

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Non-Abelian bosonization[1,2] is a remarkable equivalence between two-dimensional Dirac fermions and nonlinear σ models with a Wess-Zumino term. Like the much older Abelian bosonization, this equivalence is fruitful because two different descriptions of the same system lead to complementary insights about its behavior. For example, there are condensed-matter-physics applications to spin chains[3] and even to three-dimensional problems such as superfluid ^{3}He-A.[4] For applications to systems such as the Kondo problem or the heterotic string it would be interesting to have a similar equivalence for fermions of a single chirality. They can be dealt with, to a limited extent, by coupling to only one of the chiral components of the Dirac fermion but gravitational interactions would automatically couple to both chiralities and they will both contribute to the conformal anomaly. We would like to bosonize Weyl fermions directly. An action representing "chiral bosons" has been proposed[5] which has the right equations of motion and only one Kac-Moody algebra,[6] and is thus a candidate for a bosonic equivalent for chiral fermions. This Letter is devoted to a geometry-based demonstration that it is indeed equivalent to the fermionic theory.

Despite the apparently mysterious nature of the Fermi-Bose equivalences, it has been known for some time[7,8] that a natural geometrical approach to such relationships is provided by coherent-state path integrals. This approach is valid in any number of space-time dimensions but in more than two dimensions it replaces a few fermionic variables by a large number of commuting variables and this limits its utility. Recently Wiegman[9] has pointed out that the coherent-state construction is related to "geometric quantization"[10] and he has applied it to the case of the product of two Kac-Moody groups and to the geometric quantization of spinning particles and strings. In this paper I will show how the application of the method to a single Kac-Moody group and its representation on the space of second-quantized chiral fermions reproduces the action of Refs. 5 and 6.

First let us review the basic idea of the coherent-spate path integral for groups: Suppose $g \rightarrow D(g)$ is an irreducible unitary representation of a compact simple group G and $|0\rangle$ is the greatest-weight state in the representation. We define coherent states by

$$|g\rangle = D(g)|0\rangle . \tag{1}$$

Because of Schur's lemma these states satisfy an overcompleteness relation

$$1 = \text{const} \times \int d[g]\, |g\rangle\langle g| , \tag{2}$$

where $d[g]$ is the Haar measure.

The path-integral representation for the thermodynamic partition function

$$Z = \text{Tr}(e^{-\beta H}) , \tag{3}$$

with the trace taken over the states in the representation $D(g)$, is obtained by repeatedly inserting the resolution of the identity, Eq. (2), into the trace. We obtain formally

$$Z = \int d[g] \exp\left[\int_0^\beta (\langle g|\dot{g}\rangle - \langle g|H|g\rangle) dt\right] . \tag{4}$$

Here $d[g]$ is the path measure made from the Haar measure at each step. The integral is over all periodic paths in G taking "time" β. Actually, because of "gauge invariance" the path integral can be regarded as being over the quotient space, G/H, where H is the isotropy group of $|0\rangle$.

Part of the exponent in the integrand, $\int\langle g|\dot{g}\rangle dt = \oint\langle g|dg\rangle$, is purely imaginary and should be recognizable as the Berry phase[11] which would result if the states $|g\rangle$ were carried adiabatically around the path in the coset space G/H. Of course there is nothing adiabatic about the variations in the path integral and Berry's phase is here playing its other role as a natural connection on the Hermitian line bundle $\pi: G \rightarrow G/H$.

The action is first order in time derivatives and in Minkowski space is an example of the form

$$S = \int_{\gamma=\partial\Gamma} dt [H(x) - a(x)_\mu \dot{x}^\mu] = \int_\gamma H\, dt - \int_\Gamma \omega , \tag{5}$$

where

$$\omega = \tfrac{1}{2}\omega_{\mu\nu} dx^\mu \wedge dx^\nu, \quad \omega_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu , \tag{6}$$

leading to the classical equations of motion

$$\partial_\mu H = \omega_{\mu\nu}\dot{x}^\nu \tag{7}$$

and to Poisson brackets

$$\{f,g\} = \omega^{\mu\nu}\partial_\mu f \partial_\nu g . \tag{8}$$

Here $\omega^{\mu\nu}$ is the matrix inverse to $\omega_{\mu\nu}$ and, because of the gauge invariance, can only be defined when f,g are functions on G/H. Clearly ω plays a double role as the curvature of the line bundle and as the symplectic form defining the classical Hamiltonian dynamics.

We want to apply this formalism to the highest-weight represention of a single level-one U(N) Kac-Moody algebra[12] built out of a set of N right-going fermions. The highest-weight state in this context is just the vacuum state.

There is a nice geometrical way of regarding the vacuum and its images under the action of a loop group. This viewpoint is expounded in Ref. 13, so I shall provide only a brief description here. The one-particle Hilbert space $\mathcal{H}$ is decomposed into the spaces of positive and negative energy by the first-quantized Hamiltonian $H=-i\partial_x$ as

$$\mathcal{H}=\mathcal{H}^{(+)}\oplus\mathcal{H}^{(-)}, \tag{9}$$

and we can regard $\mathcal{H}^{(-)}$ as a point in an infinite-dimensional Grassmanian manifold, Gr($\mathcal{H}$), the rest of which consists of the images of the negative-energy subspace under the action of the loop group $\mathrm{LU}(N)=\{g(x):S^1\to \mathrm{U}(N)\}$. The highest exterior power $\Lambda^{(\max)}(\mathcal{H}^{(-)})$ of $\mathcal{H}^{(-)}$ is a one-dimensional space and its basis element is the filled Fermi sea of the second-quantized theory. The images of the vacuum, the states $|g(x)\rangle$, are the highest exterior powers of the other subspaces and are essentially Slater determinants of first-quantized states, whose wave functions are

$$\psi_n(x)=g(x)ue^{ixn} \tag{10}$$

[u is a vector in the N-dimensional defining representation of U(N)]. Further, the vacuum state, assuming that it is not degenerate (i.e., in a finite system we assume, for convenience, Neveau-Schwarz fermions), is left fixed by elements of LU(N) that are independent of x. The manifold G/H is thus LU(N)/U(N) and the states of the system form a line bundle over Gr($\mathcal{H}$) which can be identified as a set with LU(N)/U(N).[13]

With this construction the first ingredient needed for the path integral, the Berry phase which is the curvature of the line bundle, turns out[13,14] to be the pull back to LU(N)/U(N) of the first Chern class of the curvature of the bundle over Gr($\mathcal{H}$) whose fiber over a point is the subspace representing that point. It is convenient to express the curvature ω in terms of the Maurer-Cartan forms $g^{-1}\delta g$. We find that

$$\omega=\frac{1}{4\pi}\int dx\,\mathrm{tr}[(g^{-1}\delta g)\partial_x(g^{-1}\delta g)]\,, \tag{11}$$

which is the same two-form that appears as the central extension in the Kac-Moody algebra.

The second ingredient needed is the energy of the state $|g(x)\rangle=g(x)|0\rangle$. It is found to be

$$\langle g(x)|\int(-i\psi^\dagger\partial_x\psi dx)|g(x)\rangle = \frac{1}{4\pi}\int\mathrm{tr}[(g^{-1}\partial_x g)^2]dx\,, \tag{12}$$

so the complete action in our path integral is

$$S=\frac{1}{4\pi}\int\mathrm{tr}[(g^{-1}\partial_x g)^2]dx\,dt + \frac{i}{2\pi}\int\mathrm{tr}[(g^{-1}\partial_t g)\partial_x(g^{-1}\partial_\tau g)]dx\,dt\,d\tau\,. \tag{13}$$

At the moment S is apparently *not* that found in Refs. 5 and 6. We have to note that our two-form ω is not quite the same as the one found by integrating the conventional Wess-Zumino term

$$F_{\mathrm{WZ}}=\frac{1}{12\pi}\int\mathrm{tr}[(g^{-1}\delta g)^3]$$

over S^1, but it is *cohomologous* to it. However, changing the symplectic form defining a dynamical system, even by a coboundary, changes the equations of motion so we must keep track of the difference. The Wess-Zumino term can be written

$$F_{\mathrm{WZ}}=\frac{1}{4\pi}\int\mathrm{tr}([g^{-1}\partial_x g,g^{-1}\partial_t g]g^{-1}\partial_\tau g)dx\,dt\,d\tau\,, \tag{14}$$

and after some integration by parts as

$$F_{\mathrm{WZ}}=-\frac{1}{2\pi}\int\mathrm{tr}[(g^{-1}\partial_t g)\partial_x(g^{-1}\partial_\tau g)]dx\,dt\,d\tau - \frac{1}{4\pi}\int\mathrm{tr}(g^{-1}\partial_x g g^{-1}\partial_t g)dx\,dt\,, \tag{15}$$

exhibiting it as our curvature term together with a part which can be written in terms of x and t only, and so modifies the conventional action.

Using this rewrite the action becomes

$$S=\frac{1}{4\pi}\int\mathrm{tr}[(g^{-1}\partial_x g)g^{-1}(\partial_x-i\partial_t)g]dx\,dt - \frac{i}{12\pi}\int(g^{-1}dg)^3\,, \tag{16}$$

which is now the same as in Refs. 5 and 6. The equation of motion (in Minkowski space) is

$$\partial_x(g^{-1}\partial_x g+g^{-1}\partial_t g)=0\,, \tag{17}$$

with solutions

$$g(x,t)=g(x-t)h(t) \tag{18}$$

describing right-going waves on LU(N)/U(N). The arbitrary, x-independent, factor of $h(t)$ occurs because there is gauge choice involved in the projection onto the coset.

We also find that the currents $\mathrm{tr}(\lambda_i\partial_x g g^{-1})$ are conserved. They are independent of the choice of $h(t)$, so we can compute Poisson brackets by the use of the basic definition

$$\{H,f\}=\dot f \tag{19}$$

and the Hamiltonian

$$H=\int dx\,\phi^i \operatorname{tr}(\lambda_i\,\partial_x g)g^{-1}. \qquad (20)$$

The equation of motion with this Hamiltonian is

$$\dot g(x)=-2\pi\lambda_i g\phi^i(x)+gT(t)\,, \qquad (21)$$

where T is an arbitrary element of the Lie algebra of U(N). Equation (21) shows that the currents generate the left action of the loop group on the space LU(N)/U(N).

Using (19) we see that

$$\begin{aligned}\frac{d}{dt}\operatorname{tr}(\lambda_j\,\partial_x g g^{-1}) &= \operatorname{tr}[g^{-1}\lambda_j g\,\partial_x(g^{-1}\dot g)]\\ &= -2\pi\operatorname{tr}(\lambda_i\lambda_j)\partial_x\phi^i\\ &\quad +2\pi\operatorname{tr}([\lambda_i,\lambda_j]\partial_x g g^{-1})\phi^i(x)\,. \qquad (22)\end{aligned}$$

The dependence on $T(t)$ drops out, as it must, and we find the Poisson bracket provides a classical analog of the starting Kac-Moody commutation relations

$$\begin{aligned}\{J_i(x),J_j(y)\} &= i f_{ij}^k J_k(x)\delta(x-y)\\ &\quad -\frac{1}{2\pi}\operatorname{tr}(\lambda_i\lambda_j)\partial_x\delta(x-y)\,. \qquad (23)\end{aligned}$$

A further analysis[14] shows that these currents are the bosonic equivalents of the fermion currents $\psi^\dagger\lambda_i\psi$.

This Letter has concentrated on the "geometric bosonization" of a chiral fermion. The same technique can be used for Dirac fermions but is a little more complicated because a Dirac fermion is not just the product of the left- and right-going components: Although the left and right Kac-Moody algebras commute with each other and seem to be independent, the representations are linked in the same way as the mutually commuting, space-fixed and body-fixed rotations of a rigid body. Some additional constructions are required and are described elsewhere.[15]

Despite the mathematical machinery used in this Letter, no argument based on path integrals can be regarded as a rigorous demonstration of anything without much more work; all the constructions used depend in some way on smoothness assumptions about $g(x,t)$ and these are not applicable to any typical field configuration in a path integral. Despite this failing I think the coherent-state argument goes a long way to showing *why* there is a fermion-boson equivalence and why it takes the form it does. A rigorous argument can then be constructed on the lines of Ref. 1 by observing the uniqueness of the level-one representations of the Kac-Moody algebra.

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[a]Permanent address.

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