Gravitational Anomalies from the Action for Self-Dual Antisymmetric Tensor Fields in $4k+2$ Dimensions

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We show that recent actions for a single self-dual antisymmetric tensor field in $4k+2$ dimensions lead to the same one-loop gravitational anomalies as obtained a few years ago by Alvarez-Gaumé and Witten, who conjectured the necessary Feynman rules in the absence of an action principle.

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Self-dual, or anti-self-dual, antisymmetric tensor fields appear in ten-dimensional supergravity models, which themselves are the low-energy limit of string theories. In order that these supergravity theories be consistent quantum field theories, their gravitational anomalies (violations of the conservation of the energy-momentum tensor) must cancel. In particular, loops of a self-dual antisymmetric tensor, coupled to external gravitons, contain an anomaly which must cancel the anomalies due to loops with chiral spin $\frac{3}{2}$ and/or chiral spin $\frac{1}{2}$. The actions for chiral-spin-$\frac{1}{2}$ and -$\frac{3}{2}$ fields coupled to gravity are well known from supergravity theories and their anomalies have been computed straightforwardly, but for self-dual antisymmetric tensor fields recent developments concerning the construction of actions have occurred. This has enabled us to complete the proof that the gravitational anomalies cancel. Of course, the proof that they do cancel (except for the step in the proof given below) is due to work by Alvarez-Gaumé, Witten, Green, and Schwarz and led to the resurgence of string theories.

In $(d=4k+2\equiv 2n)$-dimensional Minkowski spacetime, one can impose a self-duality (or anti-self-duality) condition on a real rank-n antisymmetric tensor field

$$F_{\mu_1\cdots\mu_n}=(-g)^{1/2}(n!)^{-1}\epsilon_{\mu_1\cdots\mu_n\nu_1\cdots\nu_n}F^{\nu_n\cdots\nu_1} \equiv (*F)_{\mu_1\cdots\mu_n}, \tag{1}$$

$$F_{\mu_1\cdots\mu_n}=\partial_{\mu_1}A_{\mu_2\cdots\mu_n}+(n-1)\text{ cyclic terms}.$$

(Conventions: $\eta_{\mu\nu}=(-1,+1,+1,\ldots,+1)$; $\epsilon^{01\cdots 4k+1}=-\epsilon_{01\cdots 4k+1}=+1$; $(F^\mu,F^\nu)=[(n-1)!]^{-1}F^{\mu\nu_1\cdots\nu_{n-1}}\times F^{\nu}{}_{\nu_1\cdots\nu_{n-1}}$; square brackets on indices denote antisymmetrization with unit strength.) One may check that indeed $**F=F$ in $d=4k+2$. The one-graviton vertices for the non-self-dual case are obtained from the action

$$\mathcal{L}=-\tfrac{1}{2}(-g)^{1/2}(n!)^{-1}F_{\mu_1\cdots\mu_n}F_{\nu_1\cdots\nu_n}g^{\mu_1\nu_1}\cdots g^{\mu_n\nu_n} \equiv -\tfrac{1}{2}(-g)^{1/2}(F,F), \tag{2}$$

and are given by $\frac{1}{2}h_{\mu\nu}T^{\mu\nu}(F)$ where

$$T^{\mu\nu}(F)=(F^\mu,F^\nu)-\tfrac{1}{2}\eta^{\mu\nu}(F,F). \tag{3}$$

In the absence of an action for self-dual fields, Alvarez-Gaumé and Witten conjectured a set of Feynman rules which they used to compute the one-loop gravitational anomalies.[1] The one-graviton vertices were obtained by replacing $T^{\mu\nu}(F)$ by $T^{\mu u}(\frac{1}{2}(F+*F))$ and read

$$\mathcal{L}_{\text{int}}=\tfrac{1}{4}h_{\mu\nu}[(F^\mu,F^\nu)+(*F^\mu,F^\nu)]-\tfrac{1}{8}h^\sigma{}_\sigma(F,F). \tag{4}$$

The propagator for two non-self-dual F tensors is easily obtained from the propagator of the A fields and reads

$$\langle F_{\mu_1\cdots\mu_n}(q)F^{\nu_1\cdots\nu_n}(-q)\rangle=\frac{-i}{q^2}n!n(q^{[\nu_1}q_{[\mu_1}\delta^{\nu_2\cdots\nu_n]}_{\mu_2\cdots\mu_n]}). \tag{5}$$

From Eqs. (3)–(5) one can construct one-loop Feynman diagrams with $n+1$ external gravitons. If one uses at one vertex (4), while at the other n vertices one uses (3), and replaces at this one vertex the polarization tensor of the graviton $\epsilon_{\mu\nu}$ by $ik_\mu\epsilon_\nu+ik_\nu\epsilon_\mu$, one obtains the leading part of the covariant anomaly. [Using (4) at all vertices would yield the leading part of the consistent anomaly which is a factor $(n+1)^{-1}$ smaller.[2]]

Recently, actions for self-dual antisymmetric tensor fields have been constructed. One class of actions is manifestly Lorentz invariant, but contains a Lagrange multiplier field[3] and needs at the quantum level many extra fields.[4] Another class of actions[5–7] contains only one field $A_{i_1\cdots i_{n-1}}$ (the indices i_k will always be spacelike), both at the classical and quantum level, but these actions are not manifestly Lorentz invariant. They are, however, Lorentz invariant in the sense that one can exhibit symmetry transformations on the fields which satisfy the Lorentz algebra in flat spacetime.[5,6] Since these actions do not require Lagrange multipliers one suspects that their Feynman rules are closely related to those conjectured by Alvarez-Gaumé and Witten (AGW). In this Letter we will compute the gravitational anomalies in these theories. Although our Feynman rules differ from those of Alvarez-Gaumé and Witten, our results for the anomalies coincide with theirs.

In $d=2$ flat spacetime the action for a chiral boson is given by[5]

$$\mathcal{L}=-\dot\phi\phi'-\phi'\phi'. \tag{6}$$

The coupling to gravity was first given in Ref. 6. In Ref.

7 we derived a general algorithm to construct the coupling of $d=2$ chiral bosons to any system. Using this algorithm we obtained the coupling to supergravity as well as $d=2$ nonlinear σ models with chiral bosons.[7]

For our purposes we will need the coupling $A_{i_1\cdots i_{n-1}}$ to gravity. We begin by constructing the coupling of the field $A_{i_1\cdots i_{n-1}}$ in $d=4k+2=2n$ to gravity, extending the algorithm of Ref. 7 to higher dimensions, thus rederiving the result of Ref. 6. Decomposing $F_{\mu_1\cdots\mu_n}$ into a pure space part $F_{i_1\cdots i_n}\equiv f_{i_1\cdots i_n}$ and a timelike part $F_{\perp i_1\cdots i_{n-1}}=e_\perp{}^\mu F_{\mu i_1\cdots i_{n-1}}$, where

$$\begin{aligned} &e_\perp{}^\mu=\left(\frac{1}{N^\perp},\frac{-N^k}{N^\perp}\right),\\ &e_i{}^\mu=\delta_i{}^\mu,\\ &e_\mu{}^a \text{ inverse of } e_a{}^\mu \text{ with } a\in(\perp,i),\\ &g_{\mu\nu}=e_\mu{}^a e_\nu{}^b\tilde g_{ab},\\ &\tilde g_{\perp\perp}=-1,\ \tilde g_{ij}=g_{ij}, \end{aligned}\tag{7}$$

the nonchiral action can be written as

$$\mathcal{L}=\tfrac{1}{2}N^\perp\sqrt{g_s}[(F_\perp,F_\perp)-(f,f)],\tag{8}$$

where $g_s=\det g_{ij}$. The self-duality condition reads $F_\perp=-*_s f$ where $*_s$ is the space-dual operation, defined by

$$(*_sT)_{i_1\cdots i_{2n-1-p}}=\frac{(g_s)^{1/2}}{p!}\epsilon_{i_1\cdots i_{2n-1-p}j_1\cdots j_p}T^{j_p\cdots j_1}.\tag{9}$$

One may check that $*_s*_sf=f$. The action for self-dual fields in $d=4k+2$ can now be obtained by extending the procedure of Ref. 7. First, one writes the action

$$\begin{aligned}\mathcal{L}=N^\perp\sqrt{g_s}[(F_\perp,P)-\tfrac{1}{2}(P,P)\\ -\tfrac{1}{2}(f,f)+(\lambda,P+*_sf)],\end{aligned}\tag{10}$$

where $P_{i_1\cdots i_{n-1}}$ and $\lambda_{i_1\cdots i_{n-1}}$ are auxiliary fields with only spacelike indices. Eliminating these auxiliary fields by using their equation of motion, one is left with the action which describes, assuming a suitable falloff of the fields at spacelike infinity, a self-dual tensor:

$$\mathcal{L}=-N^\perp\sqrt{g_s}[(F_\perp,*_sf)+(f,f)].\tag{11}$$

Since $(*_sf,*_sf)=(f,f)$, this action is of the form $E\cdot B+B^2$.[6]

Note that the field $A_{0i_1\cdots i_{n-2}}$ drops out from (11). Using this action we now proceed to extract the relevant Feynman rules. The one-graviton coupling is given by

$$\mathcal{L}_{\text{int}}=\tfrac{1}{2}g_{\mu\nu}T^{\mu\nu}=h_{0i}(f^i,*_sf)+h_{ij}(f^i,f^j)-\tfrac{1}{2}h^\mu{}_\mu(f,f),\tag{12}$$

which depends only on the gauge-invariant field f. The propagator of the f field, denoted by double angular brackets, is obtained from the flat-space action for $A_{i_1\cdots i_{n-1}}$,

$$\mathcal{L}=(A,(\partial_0*_s\hat d+*_s\hat d*_s\hat d)A),\tag{13}$$

after gauge fixing by adding $\mathcal{L}_{\text{gf}}=(A,\hat d*_s\hat d*_sA)$, where $\hat d$ is the exterior derivative acting only on the space coordinates. The A-field equation from the gauge-fixed action reads

$$\left[\mathbf{q}^2\delta_{i_1\cdots i_{n-1}}^{j_1\cdots j_{n-1}}+\frac{q_0q^m}{(n-1)!}\epsilon_{mi_1\cdots i_{n-1}}{}^{j_{n-1}\cdots j_1}\right]A_{j_1\cdots j_{n-1}}(q)=0,\tag{14}$$

and the A propagator becomes

$$\begin{aligned}\langle\langle A_{i_1\cdots i_{n-1}}(q)A^{j_1\cdots j_{n-1}}(-q)\rangle\rangle=\frac{-i}{2}\frac{(n-1)!}{q^2}\Bigg[&\delta_{i_1\cdots i_{n-1}}^{[j_1\cdots j_{n-1}]}-\frac{q_0q^m}{(n-1)!\mathbf{q}^2}\epsilon_{mi_1\cdots i_{n-1}}{}^{j_{n-1}\cdots j_1}\\ &-(n-1)\frac{(q_0)^2}{(\mathbf{q}^2)^2}q^{[j_1}q_{[i_1}\delta_{i_2\cdots i_{n-1}]}^{j_2\cdots j_{n-1}]}\Bigg].\end{aligned}\tag{15}$$

The f propagator then reads

$$\langle\langle f_{i_1\cdots i_n}(q)f^{j_1\cdots j_n}(-q)\rangle\rangle=\frac{-i}{2}\frac{n!n}{q^2}\left[q^{[j_1}q_{[i_1}\delta_{i_2\cdots i_n]}^{j_2\cdots j_n]}-\frac{q_0q_m}{(n-1)!\mathbf{q}^2}q^{[j_n}q_{[i_1}\epsilon_{i_2\cdots i_n]}{}^{j_{n-1}\cdots j_1]m}\right].\tag{16}$$

Using the identity

$$q_mq^{[j_n}q_{[i_1}\epsilon_{i_2\cdots i_n]}{}^{j_{n-1}\cdots j_1]m}=\frac{\mathbf{q}^2}{n}q_{[i_1}\epsilon_{i_2\cdots i_n]}{}^{j_n\cdots j_1},\tag{17}$$

we obtain the final form of the f propagator:

$$\langle\langle f_{i_1\cdots i_n}(q)f^{j_1\cdots j_n}(-q)\rangle\rangle=\frac{-i}{2}\frac{n!n}{q^2}\left[q^{[j_1}q_{[i_1}\delta_{i_2\cdots i_n]}^{j_2\cdots j_n]}-\frac{q_0}{n!}q_{[i_1}\epsilon_{i_2\cdots i_n]}{}^{j_n\cdots j_1}\right].\tag{18}$$

The strategy we will follow to compute the anomaly originating from (12) and (18) will be to reduce the computation to the one of Ref. 1, thus showing that the anomaly is the same. To this aim we first notice an algebraic relation between the propagators (18) and (5):

$$\begin{aligned}\langle\langle f_{i_1\cdots i_n}(q) f^{j_1\cdots j_n}(-q)\rangle\rangle &= \tfrac{1}{2}\langle F_{i_1\cdots i_n}(q)(F-{*_s}F_0)^{j_1\cdots j_n}(-q)\rangle \\ &= \tfrac{1}{4}\langle (F-{*_s}F_0)_{i_1\cdots i_n}(q)(F-{*_s}F_0)^{j_1\cdots j_n}(-q)\rangle \\ &\quad + \tfrac{1}{4}\langle (F+{*_s}F_0)_{i_1\cdots i_n}(q)(F-{*_s}F_0)^{j_1\cdots j_n}(-q)\rangle \end{aligned} \tag{19}$$

($({*_s}F_0)_{i_1\cdots i_n} \equiv [(n-1)!]^{-1}\epsilon_{i_1\cdots i_n j_1\cdots j_{n-1}} F_0^{j_{n-1}\cdots j_1}$). The second term in this last identity corresponds to an insertion of the (free) equation of motion and in fact gives a nonminimal term without poles:

$$\langle (F+{*_s}F_0)_{i_1\cdots i_n}(q)(F-{*_s}F_0)^{j_1\cdots j_n}(-q)\rangle = -in!\,\delta^{[j_1\cdots j_n]}_{[i_1\cdots i_n]}\,. \tag{20}$$

Now we turn our attention to the properties of the vertices considering first the AGW formalism. The Feynman rule for the vertex used by Alvarez-Gaumé and Witten is obtained from (4),

$$R_{\mu_1\cdots\mu_n}{}^{\nu_1\cdots\nu_n} = \frac{1}{2(n-1)!}\left[\alpha^{[\nu_1}\alpha_{[\mu_1}\delta_{\mu_2\cdots\mu_n]}^{\nu_2\cdots\nu_n]} + \frac{1}{2n!}\alpha_\sigma\alpha_{[\mu_1}\epsilon^\sigma{}_{\mu_2\cdots\mu_n]}{}^{\nu_n\cdots\nu_1} + \frac{1}{2n!}\alpha^\sigma\alpha^{[\nu_1}\epsilon_\sigma{}^{\nu_2\cdots\nu_n]}{}_{\mu_n\cdots\mu_1}\right], \tag{21}$$

where we have chosen to work with a simplified polarization $h_{\mu\nu}=\alpha_\mu\alpha_\nu$ ($\alpha^2=0$), and satisfies by construction the property of self-duality:

$$R = -{*R} = -R{*}\,. \tag{22}$$

In a general one-loop graph one can split the contraction of a vertex with a propagator into two sums: one part with purely spacelike indices and another part in which the index 0 appears once; in this second sum one can then use the property (17) replacing R by $-{*R}$ (or $-R{*}$) and absorb the $*$ in the propagator. This manipulation shows that a general one-loop graph in the AGW approach can be equivalently constructed by making use of the vertices (16) with only spacelike indices, joined by the propagator

$$\langle (F-{*_s}F_0)_{i_1\cdots i_n}(q)(F-{*_s}F_0)^{j_1\cdots j_n}(-q)\rangle\,. \tag{23}$$

Now turning our attention to the theory described by the action (11) one can check that (12) gives the vertex Feynman rule

$$V_{i_1\cdots i_n}{}^{j_1\cdots j_n} = \tfrac{1}{4}R_{i_1\cdots i_n}{}^{j_1\cdots j_n}\,. \tag{24}$$

Taking into account relation (19) one can thus conclude that the relation between the (formal) expression of a general one-loop graph G computed in the theory described by (11) and G' computed in the AGW formalism is

$$G = G' + B\,, \tag{25}$$

where B are terms where one or more propagators (23) are replaced by (20). Since the propagators in (20) are independent of momenta, they are nonminimal terms which cannot contribute to the anomaly. [A similar observation was made in Ref. 1 for the propagator of bispinors, Eq. (48).] Hence the anomalies obtained from the action in (11) are the same as obtained in Ref. 1 from conjectured Feynman rules. In Ref. 1 it was explained that bosonic theories with an action principle do not contain gravitational anomalies as long as these actions are manifestly Lorentz invariant in flat spacetime. Our actions are not manifestly Lorentz invariant and this explains why they can have, and indeed do have, gravitational anomalies.

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