Conformal Invariance and the Heisenberg Model with Arbitrary Spin

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The eigenspectrum of the critical anisotropic Heisenberg model (or XXZ model), with arbitrary spin S in its disordered-ferroelectric regime $(0 \leq \gamma \leq \pi/2S)$, is solved by the Bethe-*Ansatz* method. The amplitudes of the leading finite-size corrections are calculated analytically and numerically. Using conformal invariance we give exact results for the conformal anomaly and scaling dimensions. Our results indicate that, for all spins S, the critical behavior is governed by a conformal field theory with central charge $c=1$.

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The assumption that most of the statistical-mechanics systems, at criticality, are conformally invariant has proved to be extremely fruitful in two dimensions.[1,2] The possible universality classes of critical behavior are labeled by a dimensionless number c, which is the central charge of the associated conformal (Virasoro) algebra. In the case where $c<1$ unitarity restricts[3] c to the countable set $c=1-6/m(m+1)$, $m=3,4,\ldots$. In these theories, which include the Ising $(c=\frac{1}{2})$ and the three-state Potts model $(c=\frac{4}{5})$, the operator algebra is finite and the anomalous dimensions are given by the Kac formula.[4] In the limiting case $c=1$ the algebra is not finite anymore, and in this class we have models with continuously varying exponents like the Ashkin-Teller model and the spin $S=\frac{1}{2}$ anisotropic Heisenberg model, or XXZ chain,[5] with the Hamiltonian

$$H_{1/2}=\frac{\epsilon}{4}\sum_j [\sigma_j^x\sigma_{j+1}^x+\sigma_j^y\sigma_{j+1}^y+\cos(\gamma)\sigma_j^z\sigma_{j+1}^z]\,. \quad (1)$$

Here $\epsilon=\pm 1$ and $\sigma^x,\sigma^y,\sigma^z$ are spin-$\frac{1}{2}$ Pauli matrices. This model, in the bulk limit, is gapless for $0\leq\gamma\leq\pi$ (disordered phase) with the critical exponent varying continuously[5] with γ. The operators obey a larger algebra than the Virasoro one, namely, a U(1) Kac-Moody algebra[6] with central charge $k=2S=1$.[7,8] For $\epsilon=+1$, the point $\gamma=0$ corresponds to a continuous roughening transition[9] to an ordered-antiferroelectric state, while for $\epsilon=-1$ the point $\gamma=0$ corresponds to a transition to a completely ordered ferroelectric state.[5]

The generalizations of the Heisenberg model to arbitrary spin S are related to $(2S+1)$-color-vertex models $(S=\frac{1}{2},1,\ldots)$, which are soluble by the Bethe *Ansatz*.[10,11] For $S=1$ they correspond to a nineteen-vertex model with the associated Hamiltonian[12]

$$H_1=\frac{\epsilon}{4}\sum_{m=1}^{L}\{\sigma_m-(\sigma_m)^2-2(\cos\gamma-1)(\sigma_m^{\perp}\sigma_m^z+\sigma_m^z\sigma_m^{\perp}) -2\sin^2\gamma[\sigma_m^z-(\sigma_m^z)^2+2(S_m^z)^2-2]\}\,, \quad (2)$$

where $\sigma_m=\mathbf{S}_m\cdot\mathbf{S}_{m+1}=\sigma_m^{\perp}+\sigma_m^z$ and $\sigma_m^z=S_m^zS_{m+1}^z$. Here $\epsilon=\pm1$ and S^x, S^y, and S^z are (3×3) matrices of spin 1. For general spin S, the Hamiltonian is given by a polynomial of degree $2S$ in the variables $\sigma_m^{\perp}$, σ_m^z, and S_m^z where now S^x, S^y, and S^z are spin-S matrices. These Hamiltonians can be block diagonalized into disjoint sectors labeled by $n=\sum_m S_m^z$, and in a given sector n the energies, for an L-site chain, with periodic boundaries, are given by

$$E^S\{\lambda_j\}=\epsilon\frac{\sin(2S\gamma)}{2S}\sum_{j=1}^{LS-n}\frac{\sin(2S\gamma)}{\cos(2S\gamma)-\cosh(2\lambda_j)}\,, \quad (3)$$

where $\{\lambda_j;j=1,2,\ldots,LS-n\}$ are obtained by solving the Bethe-*Ansatz* equations:[10,11]

$$\left[\frac{\sinh(\lambda_j-iS\gamma)}{\sinh(\lambda_j+iS\gamma)}\right]^L=\prod_{j\neq k=1}^{LS-n}\frac{\sinh(\lambda_j-\lambda_k-i\gamma)}{\sinh(\lambda_j-\lambda_k+i\gamma)}\,. \quad (4)$$

These spin-S Hamiltonians are massless[11] and disordered for $0\leq\gamma\leq\pi$. As in the $S=\frac{1}{2}$ case, at the isotropic limit $\gamma=0$, the model with $\epsilon=+1$ $(\epsilon=-1)$ undergoes a continuous (discontinuous) phase transition to an antiferroelectric (ferroelectric) ordered state. It was shown recently[13,14] that in the disordered-antiferroelectric regime where $\epsilon=+1$ and $0\leq\gamma\leq\pi/2S$ the critical exponents vary continuously with γ and the operators satisfy a U(1) Kac-Moody algebra with central charge $k=2S$ and conformal anomaly $c=3S/(1+S)$, for all γ.[15] In this regime the zeros of (4) corresponding to the lowest-energy state are formed by excitations above a sea of strings of size $2S$.[10,11,13]

In this Letter we study the above models in their disordered-ferroelectric regime $\epsilon=-1$ and $0\leq\gamma\leq\pi/2S$ and, as we shall see, a very different behavior occurs. Solving (4) numerically we verify that, contrary to the $\epsilon=1$ case, in this regime the roots $\{\lambda_j\}$ corresponding to the ground state are formed by a sea of antiparticles[16] $\{\lambda_j\}=\{\lambda_j^R+i\pi/2\}$, $j=1,2,\ldots,Ls-n$, where λ_j^R are real numbers. The lowest energies correspond to excitations above this sea. Using this fact the density of zeros of (4) in the bulk limit is given by[17]

$$\sigma_\infty^S(\lambda)=\frac{1}{\pi-\gamma}\sum_{s=-(S-\frac{1}{2})}^{S-\frac{1}{2}}\frac{\cos[\gamma\pi s/(\pi-\gamma)]\cosh[\lambda\pi/(\pi-\gamma)]}{\cos[2\gamma\pi s/(\pi-\gamma)]+\cosh[2\lambda\pi/(\pi-\gamma)]}, \quad (5)$$

and the ground-state energy per particle, in this limit, is[17]

$$e_\infty^S=-\frac{\sin(2S\gamma)}{4S}\int_{-\infty}^{+\infty}\frac{\sinh^2(S\gamma X)}{\sinh(\gamma X/2)\cosh[X(\pi-\gamma)/2]\sinh(X\pi/2)}dX. \quad (6)$$

We also calculated their dispersion relation, whose low-momentum behavior permitted us to extract the sound velocity,

$$\xi_S=\pi\sin(2S\gamma)/2(\pi-\gamma), \quad 0\le\gamma\le\pi/2S. \quad (7)$$

The conformal anomaly c can be estimated[18] from the large-L behavior of the ground-state energy $E_0^S(\gamma,L)$ of the L-site Hamiltonian,

$$E_0^S(\gamma,L)/L=e_\infty^S-\pi\xi_S c/6L^2+o(L^{-2}). \quad (8)$$

In the regime we are studying, the Bethe-*Ansatz* equations can be transformed in a set of real equations and the eigenenergies can be calculated analytically using standard methods.[19] Our results, using (6)-(8) give us $c=1$ for arbitrary γ and spin S. We also have solved (4), for finite L, for several values of γ and S and in Table I we show some of the estimates for the conformal anomaly, together with their exact results. It is important to notice here that although the signal of ϵ in the spin-S Hamiltonians does not change their explicit symmetries, the universality classes change remarkably.[13]

The conformal invariance of the critical system also gives us a powerful way to calculate the scaling dimensions of operators, governing the criticality, from the eigenspectrum of the finite-chain Hamiltonian.[2] For each operator, with scaling dimension x and spin s, there exists a tower of states in the finite-size-L Hamiltonian. In the case of a periodic boundary condition, the energy and momentum of these states are given by[2]

$$E_{M,\bar{M}}=E_0^S(L)+2\pi\xi_S(x+M+\bar{M})/L+o(L^{-1}),$$
$$P_{M,\bar{M}}=2\pi(s+M-\bar{M})/L;\quad M,\bar{M}=0,1,2,\ldots, \quad (9)$$

respectively. From the large-L dependence of the eigenspectrum, which we calculated analytically,[17] we obtain through (9) the exact scaling dimensions and spins of several primary operators, together with their conformal towers.[20].

Let us discuss initially the case of L even and a periodic boundary condition. The distribution of the zeros of (4) corresponding to the lowest-energy state in the sector n is formed by a sea of $LS-n$ antiparticles.[16] These states are related, through (9) to the spinless operators $O_{n,0}$ with dimension $x_{n,0}=n^2x_p$, where $x_p=\gamma/2\pi$. The operators $O_{1,0}$ and $O_{2,0}$ correspond to the polarization and energy operators in the eight-vertex model ($S=\frac{1}{2}$).[5] The distribution of roots of the Bethe-*Ansatz* equations (4) for the excited states, on sector n, are formed by excitations of real and stringlike particles (or both) in a sea of antiparticles and they are related with the operators $O_{n,m}$ with dimensions

$$x_{n,m}^S=n^2x_p+m^2/4x_p, \quad n,m=0\pm1,\pm2,\ldots, \quad (10)$$

and spin $s_{n,m}=nm$. It is interesting to remark here that the dimensions $x_{n,m}^S$ do not depend on the spin S of the model. The above operators $O_{n,m}$ are the analogs of the Gaussian-model operators[21] composed of a spin-wave excitation of index n and vorticity m, in perfect agreement with known results for the spin $S=\frac{1}{2}$ model.[7,8] We also calculate dimensions of primary operators which are constant in the whole range $0\le\gamma\le\pi/2S$. These are, for example, a spin-1 operator with dimension $x=1$ and a spinless marginal operator (dimension $x_{\text{mar}}=2$). This last operator governs the motion along the critical line[21] and corresponds in the $S=\frac{1}{2}$ case to the four-spin coupling of the eight-vertex model. To illustrate these results we show in Table II the finite-size estimates for the

TABLE I. Finite-lattice estimates for the conformal anomaly of the spin-S model: $c_L^S(\gamma)=[e_\infty^S-E_0^S(\gamma,L)/L]6L^2/\pi\xi_S$.

L	$c_L^1(\pi/3)$	$c_L^{3/2}(\pi/4)$	$c_L^2(\pi/5)$	$c_L^{5/2}(\pi/6)$	$c_L^3(\pi/7)$
8	0.992 190	0.999 197	1.000 387	1.000 953	1.001 284
28	0.999 423	0.999 928	1.000 031	1.000 077	1.000 104
48	0.999 804	0.999 975	1.000 010	1.000 026	1.000 035
68	0.999 902	0.999 988	1.000 005	1.000 013	1.000 018
88	0.999 941	0.999 993	1.000 003	1.000 008	1.000 010
100	0.999 954	0.999 994	1.000 002	1.000 006	1.000 008
Exact	1.0	1.0	1.0	1.0	1.0

TABLE II. Finite-lattice estimates of the scaling dimensions of the spin-S model: $x_{n,m}^{S}(\gamma) = n^2 x_p + m^2/4x_p$, $x_p = \gamma/2\pi$. $x_{\rm mar}^{S}$ corresponds to the marginal operator.

L	$x_{0,1}^{1}(\pi/3)$	$x_{\rm mar}^{1}(\pi/3)$	$x_{0,1}^{3/2}(\pi/4)$	$x_{1,0}^{2}(\pi/5)$	$x_{2,0}^{5/2}(\pi/6)$
8	1.570838	1.967905	2.074452	0.099712	0.333508
28	1.505985	1.997448	2.005670	0.099976	0.333348
48	1.502040	1.999133	2.001922	0.099992	0.333338
68	1.501017	1.999568	2.000956	0.099996	0.333336
88	1.500607	1.999742	2.000442	0.099998	0.333335
100	1.500470	1.999800	2.000442	0.099998	0.333334
Exact	1.5	2.0	2.0	0.1	$0.\dot{3}$

scaling dimensions of several operators for some values of γ and spin S. All these results indicate that the critical behavior of the disordered-ferroelectric regime of the spin-S Heisenberg model is described in terms of Gaussian fields $\phi_{\Delta^+,\Delta^-}^{n,m}$ with dimension $x_{n,m} = \Delta^+ + \Delta^-$ and spin $s_{n,m} = \Delta^+ - \Delta^-$, where $\Delta^{\pm} = (n\sqrt{x_p} \pm \frac{1}{2} m\sqrt{x_p})^2/2$. These Gaussian fields satisfy a U(1) Kac-Moody algebra[6,22] with central charge $k = 1$, for all spin-S models, contrary to the critical behavior of the disordered-ferroelectric regime, where the central charge is $k = 2S$.[15] We have also calculated the large-L corrections to (8) and (9); for example, the leading corrections to (9) for the dimensions $x_{n,0}$ are of the form $L^{-2}(a_0 L^{-2(\pi-\gamma)/\gamma} + a_1 L^{-2})$. All the corrections we calculated can be accounted for by using perturbation theory and the fact that the Hamiltonian deviates from that of the continuum theory by irrelevant operators.[2] As in the $S = \frac{1}{2}$ case[7] these corrections can be explained as arising from the irrelevant operator, with dimension $x = 4$, of the conformal block of the identity operator and the primary operator $O_{2,0}$ with dimension $x_{0,2} = 2\pi/\gamma$.

In the case where the lattice size L is an odd number the sectors are labeled by $n = 0, \pm 1, \pm 2, \ldots$ if S is an integer and $n = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots$ if S is a half integer. The scaling dimensions obtained by using in (4) the ground state of the $(L+1)$-size system are also given by $x_{n,m}$.

For the spin-$\frac{1}{2}$ system, it was shown[7,23] that all the minimal models ($c < 1$) can be obtained by one's changing continuously the boundary condition compatible with the U(1) symmetry. This fact motivated us to analyze our spin-S system with the toroidal boundary condition specified by the angle ϕ ($0 \le \phi \le 2\pi$):[24]

$$S_{L+1}^x \pm i S_{L+1}^y = e^{\pm i\phi}(S_1^x \pm i S_1^y), \quad S_{L+1}^z = S_1^z. \quad (11)$$

The Bethe-*Ansatz* equations (4), for periodic boundary conditions, can be generalized to include these cases.[13] We obtain, by solving these equations analytically and numerically, the finite-size corrections of the eigenspectrum which would correspond[23] to a periodic critical model with conformal anomaly $c_\phi^S = 1 - 3(\phi/\pi)^2/2\gamma$ and the scaling dimension $x_{n,m+\phi/2\pi}$ given in (10), for all spins S. Choosing properly[23] the angle ϕ, we can obtain the conformal anomaly and operators corresponding to the minimal series[3] ($c < 1$).

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[16]We denote by antiparticle a root λ of the Bethe-*Ansatz* equations with imaginary part $\pi/2$. Also from (3) and (4) the roots and energies in regions $0 \leq \gamma \leq \pi/2S$ and $\pi(1-1/2S) \leq \gamma \leq \pi$ are exactly related. For S an integer, the roots and energies are the same while for S a half integer, the roots $\{\lambda_i\}$ in both regions differ by $i\pi/2$ and the energies by a minus sign.

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