Experimental Realizations of Kicked Quantum Chaotic Systems

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We propose definite experiments to provide realizations of a number of important models in the field of quantum chaos. These models share the feature that they are driven impulsively in time, i.e., kicked. In quantum language, a kick drives the system from one energy level to a neighboring one. The most fundamental question is whether localization occurs, i.e., whether eventually the system is driven far from its initial level or not. The most straightforward realizations are in optical fibers where the language of modes and propagation constants replaces quantum terminology.

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Quantum chaos[1,2] is the study of systems described by wave equations under the following conditions: (1) The system is Hamiltonian, i.e., little or no dissipation and noise; (2) low dimension; (3) low symmetry; and (4) wavelength rather short. There are many interesting systems of scientific and even engineering importance whose theory falls under this definition. An example is the highly excited vibrational and rotational states of small molecules, described by Schrödinger's equation. Another is the propagation of plasma waves in devices such as a tokamak, described by magnetohydrodynamic equations. The closely related case of randomness and disorder is usually considered separately.

However, the current level of sophistication is so low that most theories are restricted to cases of phase space dimension 3 or 4. Unfortunately, very few realizable examples of conserving systems with such low effective dimensionality are known. (For example, orbits in the three-body problem lie on a phase-space manifold of dimension 8.) Essentially the single, and very beautiful, experimental example is Rydberg hydrogen driven by a strong microwave field,[3,4] whose phase-space dimension seems to be 3. Another type of experiment measures just the spectrum and not the dynamics. Hydrogen in a magnetic field is most fruitful is this case.[5]

In fact, however, most theoretical work is on still more specialized cases[6–8] in which the system propagates in free space except for kicks or reflections at sharp time or space coordinates. Numerical (and analytic) results are much easier to obtain in these cases, and the results are believed to be qualitatively generic.[4,9] It is the purpose of this note to suggest experimental realizations of these theoretically most tractable cases, something which up to now has not been achieved in practice and for which there are very few concrete proposals.

Because the wavelength is short, the eikonal approximation is suggested. This approximation reduces the difficult partial differential equations to much more tractable ordinary, but generally nonlinear, differential equations, ODE's; e.g., Newton's equations replace the Schrödinger equation. Because of nonlinearity and low symmetry these ODE's exhibit chaotic solutions, if the dimension of phase space is 3 or greater. It remains a major problem to find and classify the spectrum, eigenstates, and other wave properties of such systems, even with the help of the numerical solutions of the ODE's.

For the most part we restrict our proposals to the following system: Propagation of light in optical fibers or thin plates.[10,11] There have been tremendous technological advances in this field in the last couple of decades which make very sophisticated experiments possible. Light can propagate many meters in practical fibers without change of mode or loss of amplitude. We here show that very interesting experiments are feasible.

We confine attention to the widely used and practical case of *weak guidance*. The main advantage of weak guidance is that polarization effects are small; i.e., the vector Maxwell's equations may be replaced by the scalar wave equation. This is essentially because the paraxial approximation is also good; the light propagates almost parallel to the fiber axis. Weak guidance means that $\Delta \equiv (n_{\rm co}^2 - n_{\rm cl}^2)/2n_{\rm co}^2$ is small (typically 10^{-2}). Here n is the index of refraction, assumed real, and the subscripts refer to the core and cladding. To be concrete, one may keep in mind a circular core of radius ρ or a slab with core thickness 2ρ, and a step profile in the dielectric constant. The cladding is assumed to be infinitely thick, a good approximation. However, graded profiles are also very interesting. We adhere to the notation of Snyder and Love.[10]

We want a many-mode fiber (transverse wavelength $\ll \rho$), thus with large "fiber parameter" $V \equiv k\rho n_{\rm co}(2\Delta)^{1/2}$, where $k = 2\pi/\lambda = \omega/c$ is the free-space wave number. The number of bound modes is $4V/\pi$ for the planar case, and $V^2/2$ for the circular case, counting the two polarizations. Under these conditions, geometrical optics (the eikonal approximation) is usually adequate.[10] In that language, bound modes impinge on the core-cladding boundary at a glancing angle ϑ giving total internal reflection $[\vartheta < \vartheta_c = (2\Delta)^{1/2}]$.

The equation, $\nabla^2+n^2k^2=0$, satisfied by a function Ψ (say the electric field) is rewritten

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial z^2}-\frac{1}{2}\nabla_t^2+\mathcal{U}_0(r)+\epsilon\mathcal{U}_1(r,z)\right]\Psi=\tfrac{1}{2}\kappa^2\Psi\,. \quad (1)$$

We have set the refractive-index profile as $n^2(r,z)=n_0^2(r)+n_1^2(r,z)$, where $n_0^2(r)$ depends only on the transverse direction(s) r, and the "perturbative" part, n_1^2, depends on both z and r. Then $\mathcal{U}_0=\frac{1}{2}[\kappa^2-k^2n_0^2]$ and $\epsilon\mathcal{U}_1=-\frac{1}{2}k^2n_1^2$. Here ∇_t is the gradient transverse to the fiber, and $\kappa=kn_{\rm cl}$ is the wave number in the cladding. A time dependence $\exp(-i\omega t)$ is assumed. The strength parameter ϵ is separated out for convenience. We have departed from Ref. 10 to make Eq. (1) look like the (mass 1) Schrödinger equation.

Neglecting $\mathcal{U}_1$, and putting $\Psi=e^{i\beta_l z}\varphi_l(r)$, where β_l is the propagation constant for the lth mode, gives an eigenvalue equation for φ_l,

$$[-\tfrac{1}{2}\nabla_t^2+\mathcal{U}_0(r)]\varphi_l\equiv\mathcal{H}_0\varphi_l=E_l\varphi_l\,, \quad (2)$$

where $E_l=\frac{1}{2}(\kappa^2-\beta_l^2)\cong\kappa(\kappa-\beta_l)$ and $U_l=\rho[2k^2n_{\rm co}^2\Delta+2E_l]^{1/2}$ is the core parameter, or dimensionless transverse wave number. Bound modes have negative "energy" E_l, or $U_l<V$.

We now consider the perturbation $\epsilon\mathcal{U}_1$. We take it to be periodically bumped; i.e., $\mathcal{U}_1$ vanishes except within a short distance ζ of $z=jZ$, where j is an integer. [It is also interesting to space the pulses randomly or quasiperiodically, etc.] Letting $\Psi=e^{i\kappa z}\varphi(r,z)$, an approximating $-\partial^2/\partial z^2\cong\kappa^2-2i\kappa\partial/\partial z$ we have the equation,

$$i\kappa\frac{\partial\varphi}{\partial z}=[\mathcal{H}_0+\epsilon\mathcal{U}_1(r,z)]\varphi\,, \quad (3)$$

i.e., the periodically pulsed "time dependent" Schrödinger equation. Neglect of the second z derivative of φ is tantamount to the neglect of reflection from the bumps in the dielectric constant. [The matrix element for such a reflection is $\int dz\, e^{2i\kappa z}\mathcal{U}_1(r,z)$. Suppose the z dependence of $\mathcal{U}_1$ to be smooth, say $\mathcal{U}_1=\kappa^2\mathcal{V}(r)W(z/\zeta)$, where for example $W(s)=\exp(-s^2)/\sqrt{\pi}$ or $1/(\pi\cosh s)$. If $\kappa\zeta\gg1$ we can neglect reflection.]

We next consider the condition under which the bump effectively acts like a δ-function kick. The formal solution for the "time" evolution operator $\mathbf{U}(z,z')$ [$\varphi(z)=\mathbf{U}(z,z')\varphi(z')$] is given by

$$\mathbf{U}(z,z')=\left(\exp\left[-\frac{i}{\kappa}\int_{z'}^{z}dz_1\,\epsilon\tilde{\mathcal{U}}_1(z_1)\right]\right)_+\,, \quad (4)$$

where the "+" indicates the earliest "times" are ordered to the right. Here $\tilde{\mathcal{U}}_1=\exp(i\mathcal{H}_0z/\kappa)\mathcal{U}_1\exp(-i\mathcal{H}_0z/\kappa)$. A typical term in the expansion of $\mathbf{U}$ is

$$\left(\frac{\epsilon}{i\kappa}\right)^N\int_{z'}^{z}dz_1\int_{z'}^{z_1}dz_2\cdots\int_{z'}^{z_{N-1}}dx_N\,\tilde{\mathcal{U}}_1(z_1)\cdots\tilde{\mathcal{U}}_N(z_N)\,.$$

Taking a matrix element between φ_f and φ_l and assuming a smooth z dependence gives a sum of terms with factors $\mathcal{V}_{jm}\exp[-iz(E_j-E_m)/\kappa]W(z/\zeta)$, where $\mathcal{V}_{jm}$ is the matrix element of $\mathcal{V}$ between transverse modes. If the energy dependence $\exp[-iz(E_j-E_m)/\kappa]$ can be neglected, then the time ordering can be ignored, and the traversal of the kick is given by (4) with no "+," which is just the formula for the propagation through a δ-function kick. The condition is thus $\zeta(E_j-E_m)/\kappa\cong\zeta(\beta_m-\beta_j)\ll1$ for the largest $\beta_m-\beta_j$ with m,j connected by appreciable matrix elements. Assuming the matrix elements connect only nearby modes, $j\cong m\pm1$, the largest difference for a step-profile planar structure or fiber will be near cutoff, i.e., for j as large as possible. The step-profile fiber has, crudely, $E_j\cong\pi^2j^2/8\rho^2-\kappa^2\Delta$. The condition is then $\pi\zeta(2\Delta)^{1/2}/2\rho\ll1$, which is satisfied for $\zeta\simeq\rho$. Blümel, Fishman, and Smilansky[9] have discussed the case in which this condition is not strongly satisfied and they find that the many results are insensitive to it. At the opposite extreme, for $\zeta(\beta_m-\beta_j)\gg1$, for the smallest modal differences, we would have no transitions between modes, and the adiabatic approximation would be good.

This result gives for the evolution operator from just before one kick to just before the next (a quantum map[2]),

$$\mathbf{U}=\exp[-i\mathcal{H}_0Z/\kappa]\exp[-i\epsilon\kappa\zeta\mathcal{V}]\,. \quad (5)$$

For a step profile, $\mathcal{H}_0Z/\kappa$ can be "quantized," approximately as $(\hbar^2\pi^2j^2/8-Z^2\Delta/\rho^2)/\hbar$ where the dimensionless "Planck's constant" $\hbar\equiv Z/\rho^2\kappa=\lambda Z/2\pi\rho^2n_{\rm cl}$. The second exponential can be written $\exp[-iK\mathcal{V}/\hbar]$, where K is the "classical" kicking strength given by $K=\epsilon\zeta Z/\rho^2$. This is a "two parameter" system much studied by quantum chaos theorists,[6-8] where K is "classical," independent of λ, parametrizing chaos, and $\hbar$ parametrizes finite-wavelength effects.[12] ($Z^2\Delta/\rho^2$ is a second "classical" parameter characterizing the depth of the potential, but other dimensionless numbers, e.g, Δ, Z/ρ, and ζ/Z do not appear independently in the problem.) The greatest interest attaches to the cases where both K and $\hbar$ are of order unity. A typical operating condition would be $\lambda\cong0.5-1.5$ μm, $\rho\cong100\lambda$, $\zeta\cong\rho$, and $Z\cong10^3\rho$. Under these circumstances K can be made of order 1 if $\epsilon\cong10^{-3}$.

The question then is how one can change the dielectric constant locally in this fashion. One possibility is to build it in during the drawing of the fiber by varying the composition. This has the disadvantage that K cannot then be readily varied during the course of an experiment. (In the planar case, one could vary the angle of propagation so that Z and ζ would effectively be varied together. This is likely to be impractical for planar optical systems but is suitable for channeling systems to be mentioned.) There are many possibilities. Among them we have considered inducing local strains by the pressure of a knife edge, inducing local changes in carrier concentration by attaching a gate, exploiting nonlinear optics effects, and using piezoelectric materials which can be manipulated by an electric field.

The most convenient way we have found so far is to induce the bumps by local heating either electrically or by blackening the surface at the bump and heating it with radiation. In these cases the profile in the z direction will be smooth and $\zeta \cong \rho$. If a steady-state temperature gradient is maintained across a bump in the planar case, then $\int \mathcal{U}_1 dz \propto x$. This "dipole" coupling is often studied in modeling "atoms" driven by a "microwave" field. This is the only nontrivial steady-state possibility, since the integral of the local temperature $\int T dz$ satisfies the one-dimensional Laplacian. It should be easy to use transient heating, however, in view of the difference in time scale between optics and heat diffusion. Then $\int T dz$ will go approximately as x^2 in the simplest case, i.e., $\epsilon \cong \frac{1}{4}\rho^2 T'' d\ln n^2/dT$, $\mathcal{V} = x^2/\rho^2$. This is another case which is much studied. Since $dn/dT \cong -0.6 \times 10^{-5}/°\mathrm{C}$ (for fused silica), one must achieve a transient temperature difference between the axis and boundary of the fiber of order 50 °C.

For the circular fiber, in the case in which the bumps are formed by transient heating in an axially symmetric manner, the perturbation will be approximately r^2. This perturbation does not change the angular dependence of the light. Thus the $HE_{1m}(l=0)$ modes for example are not mixed with the other modes. Again, a kicked one-dimensional system results, which is classically identical to the planar system kicked by an x^2 potential. We shall see that the subspace of $l \neq 0$ modes essentially differs from that of $l=0$. Heating with a steady-state flow of heat transverse to the fiber will give a perturbation $\propto r \sin\vartheta$, which couples both radial and angular modes and gives rise to a two-dimensional kicked system.

The cases of kicking a square well with either an x or an x^2 potential are special. These cases have classically the same Lyapunov exponents as for kicking in free space,[8] since reflection at a sharp boundary changes the global orbit but does not affect its local properties. The free-space problems can be solved analytically.[13,14] It will, of course, be of much interest to realize these well studied problems experimentally, if only to verify that the experimental setup is working. Kicking the one-dimensional $l \neq 0$ subspace in the circular fiber *is* generic, however, since the effective one-dimensional potential includes the term l^2/r^2. One may equally use graded fibers to achieve generic kicked systems.

Even more things can be measured in fibers than in the driven Rydberg hydrogen experiment.[3] Light in a given mode incident on the train of bumps will undergo deflections by the bumps; in other words, it suffers transitions between modes. If the total deflection becomes large enough, the light exceeds the critical angle, corresponding to the atom ionizing. If the light remains in the bound modes, it is possible to measure the distribution in these modes. Important issues in the hydrogen case such as smooth versus sudden turn-on of the bumps can also be explored.[15]

A standard theoretical calculation is for the mean "energy" $\langle E_n \rangle$ vs "time" (=number of kicks). This is just the mean-square angle of light propagation. It is expected to follow the geometrical-optics prediction up to some "break time" depending on $\hbar$, and further spread in angle is suppressed by finite-wavelength "localization" effects, in the generic case. This localization is known to be a direct analog of Anderson localization in random conductors.[6] It should be possible in fibers (but not in hydrogen) to make measurements directly verifying this prediction. Figures 5 and 6 of Ref. 8, for example, might be realized, but similar figures occur in many references.[16] One may also find "quantum resonance" effects at rational values of $\pi\hbar$, e.g., $\hbar = 2/\pi$. This corresponds to $Z(\beta_{j+s} - \beta_j) = 2\pi \times$integer for many j, which increases the mode number much faster than the geometrical-optics prediction.

Many variations on these experiments are of interest. Using graded profiles, e.g., triangular,[17,18] parabolic,[2,14] and if possible[19] $1/r$, would make contact with other efforts. Different results are expected for the same total kicking strength if ϵ is a function of kick number n. Much fascination attaches[20,21] to the case that $\epsilon(n) = \epsilon_0$ plus a periodic or random function. Similar modulation of the spacing Z can be done. Noise or dissipation can be deliberately added to the system. In all of these cases the theory can be more or less readily worked out, at least numerically, and key aspects of the theoretical understanding will be put into practice.

More challenging to the theorist will be the case mentioned above where the modes are driven through a two-dimensional mode space. Here there are some general arguments of localization theory available but only very preliminary actual model calculations.[20,22] Using fibers of different cross section, e.g., square or elliptical, will be of interest. Finally, there is the possibility of using a kicking potential which mixes polarizations. This will allow a detailed experimental study of quantum chaos and localization phenomena[23] in four dimensions, a very important first.

It may also be possible to make perturbations which reflect the light appreciably or ones which are too long and smooth to be considered to be Δ functions. Instances of the latter case have been treated with mathematical rigor[24] and localization is proved. However, the "ray" prediction also gives localization and differs little from the full theory. The former case is not too different from the one treated here, for periodic bumps, but cannot be directly compared with the existing theory of the Schrödinger equation.

We also mention that a theoretically very similar system is channeling,[25] for example, of relativistic (100 MeV) electrons or positrons in silicon crystals. Of course, the experimental milieu is totally different, since the length scale is a millionfold shorter for channeling. These projectiles are confined for long distances to chains or planes of atoms in the crystal. Changing the energy changes the relativistic mass (not the speed) which enters into the dimensionless Planck's constant. Growing a crystal with regularly spaced planes with additional doping (a superlattice) gives the kicking. One may change the effective spacing Z of the planes by channeling at an angle, thus giving a handle on a second parameter. It is not quite as easy to achieve small noise and dissipation in channeling as in fibers and one has less control over the shape of the effective potential holding the projectiles in the channel. Nevertheless, channeling technology is rather advanced and interesting experiments in quantum chaos could be done.

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[1]For a review see, e.g., B. Eckhardt, Phys. Rep. **163**, 205 (1988).

[2]M. V. Berry, in *Chaotic Behavior of Deterministic Systems,* Proceedings of Les Houches Summer School, Session XXXVI, edited by G. Iooss, R. H. G. Helleman, and R. Stora (North-Holland, Amsterdam, 1983).

[3]J. E. Bayfield and D. W. Sokol, Phys. Rev. Lett. **61**, 2007 (1988); E. J. Galvez, B. E. Sauer, L. Moorman, P. M. Koch, and D. Richards, Phys. Rev. Lett. **61**, 2011 (1988).

[4]R. Blümel and U. Smilansky, Phys. Rev. Lett. **58**, 2531 (1987); S. M. Susskind and R. V. Jensen, Phys. Rev. A **38**, 711 (1988); J. G. Leopold and D. Richards, Phys. Rev. A **38**, 2660 (1988); G. Casati, B. V. Chirikov, D. L. Shepelyansky, and I. Guarneri, Phys. Rep. **154**, 77 (1987), and references therein.

[5]J. Main, G. Widbusch, A. Holle, and K. H. Welge, Phys. Rev. Lett. **57**, 2789 (1986); Dieter Wintgen, Phys. Rev. Lett. **61**, 1803 (1988).

[6]S. Fishman, D. R. Grempel, and R. E. Prange, Phys. Rev. Lett. **49**, 509 (1982); Phys. Rev. A **29**, 1639 (1984).

[7]T. Hogg and B. Huberman, Phys. Rev. Lett. **48**, 711 (1982); Phys. Rev. A **28**, 22 (1983).

[8]A. Cohen and S. Fishman, Int. J. Mod. Phys. B **2**, 103 (1988).

[9]B. Blümel, S. Fishman, and U. Smilansky, J. Chem. Phys. **84**, 2604 (1986).

[10]A. W. Snyder and S. D. Love, *Optical Waveguide Theory* (Chapman and Hall, London, New York, 1983).

[11]D. Marcuse, *Theory of Dielectric Optical Waveguides* (Academic, New York, 1974).

[12]A different optical analog was suggested by J. Krug, Phys. Rev. Lett. **59**, 2133 (1987).

[13]R. Blümel, R. Meir, and U. Smilansky, Phys. Rev. **103A**, 353 (1984).

[14]M. V. Berry, N. L. Balazs, M. Tabor, and A. Voros, Ann. Phys. (N.Y.) **122**, 26 (1979).

[15]R. V. Jensen, Phys. Scr. **35**, 668 (1987), and references therein.

[16]See, e.g., D. L. Shepelyansky, Physica (Amsterdam) **8D**, 208 (1983); S. Fishman, D. R. Grempel, and R. E. Prange, Phys. Rev. A **36**, 289 (1987).

[17]E. Shimshoni and U. Smilansky, Nonlinearity **1**, 435 (1988).

[18]M. Sherwin (private communication on experiments in progress).

[19]R. Blümel and U. Smilansky, Phys. Rev. Lett. **52**, 137 (1984); Phys. Rev. A **30**, 1040 (1984).

[20]G. Casati, I. Guarneri, and D. L. Sheplyansky, Phys. Rev. Lett. **62**, 345 (1989).

[21]U. Sivan and A. Saar, Europhys. Lett. **5**, 139 (1988).

[22]E. Doron and S. Fishman, Phys. Rev. Lett. **60**, 867 (1988).

[23]For a review on localization theory, see, e.g., P. A. Lee and T. V. Ramakrishnan, Rev. Mod. Phys. **57**, 287 (1985).

[24]J. Bellissard, in *Trends and Developments in the Eighties,* edited by S. Albeverio and Ph. Blanchard (World Scientific, Singapore, 1986).

[25]S. T. Picraux, W. R. Allen, R. M. Biefeld, J. A. Ellison, and W. K. Chu, Phys. Rev. Lett. **54**, 2355 (1985).