Confined States in Phase Dynamics

Helmut R. Brand[(1,2)] and Robert J. Deissler[(2,3)]

[(1)]*Fachbereich 7, Physik, Universität Essen, D 4300 Essen 1, West Germany*

[(2)]*Center for Nonlinear Studies, MS-B 258, Los Alamos National Laboratory, University of California, Los Alamos, New Mexico 87545*

[(3)]*National Center for Atmospheric Research, Boulder, Colorado 80307-3000*

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We discuss the possibility of the coexistence of two states with different wavelengths in the framework of phase dynamics. We investigate in detail this phenomenon which is intrinsically nonlinear and give a Lyapunov functional for the case of a phase associated with a stationary pattern. The relationship of the confined states discussed here with recent experimental observations of localized states possessing a wavelength different from that in the bulk of the container for the case of slot convection and for the Taylor instability between co-rotating cylinders is critically examined.

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Recently the observation of confined states in an annulus near the onset of convection in binary-fluid mixtures has been reported.[1] By confined states one means that for part of the annulus a convective pattern is visible—in the case of binary mixtures, traveling rolls —whereas apparently the rest of the convective cell is free of convection. This phenomenon, which was not predicted theoretically, has since attracted considerable interest.[2-5]

Here we predict that an analogous phenomenon exists in phase dynamics, namely the coexistence of two patterns showing different wavelengths in different parts of the cell. By phase dynamics we mean here the analog of hydrodynamics for large-aspect-ratio pattern-forming nonequilibrium systems.[6-8] The analogs of the hydrodynamic variables are the phase variables, whose slow spatial and temporal variations characterize the changes of the wavelength of the pattern as a function of space and time. We point out that confined states in phase dynamics are an intrinsically nonlinear phenomenon, which cannot be obtained from linearized phase equations. We critically compare our predictions with recent experimental results on slot convection in a simple fluid[9] and on the Taylor instability for the flow in the gap between co-rotating cylinders.[10,11] By slot convection one means that the height of the cell is larger than the width whereas the length is large compared to both. Critical experiments to test the applicability of our approach to the systems already studied experimentally, as well as to other systems, are suggested.

The importance of studying the slow spatial and temporal evolution of a pattern with a characteristic wavelength has been recognized in Ref. 6 and it has been suggested that a diffusion equation for the variation of the wavelength,

$$\dot{\psi} = D\psi_{xx} , \tag{1}$$

is obtained even well above onset of the instability, a regime that is not accessible by amplitude and envelope equations. In Eq. (1) we have written down the one-dimensional special case of the equation derived in Ref. 6 for Bénard convection. From the equation one reads immediately that local perturbations in the average wavelength diffuse as a function of time and do not propagate. Close to local thermodynamic equilibrium diffusive modes are known to occur, for example, for heat and vorticity diffusion in a simple liquid. The applicability of the concept of phase diffusion has been experimentally verified quantitatively for the case of Bénard convection in Ref. 12. Since then a large number of studies on phase dynamics have appeared. Most of those, however, have concentrated on phenomena obtainable from linearized phase equations[6,13,14] or on the theoretical investigation of the Kuramoto-Sivashinsky (KS) equation,[15-17] which is known to lead to weak turbulence for negative values of the diffusion coefficient.[18]

Here we study in detail a phenomenon which is intrinsically nonlinear and which can occur, e.g., for stationary patterns such as stationary Bénard rolls. To focus on the essential features of the nonlinear results predicted here, we investigate exclusively spatial variations in one direction. That is, we assume, for example, for the case of the Bénard instability that the width of the cell is small compared to its length and thus that instabilities parallel to the roll direction are suppressed.

For such a pattern one has only one phase variable ψ describing the spatial and temporal variations of the wavelength,

$$\dot{\psi} = [D + E\psi_x + F(\psi_x)^2]\psi_{xx} - G\psi_{xxxx} , \tag{2}$$

where we have kept the cubic nonlinearity in Eq. (2) to guarantee that the solution is bounded. As has been pointed out in Ref. 7 one has to lowest order in the nonlinearity only the quadratic term, which does not lead, however, to bounded solutions (Ref. 7).

First we note that Eq. (2) can also be written in terms of the wave vector $\psi_x = q$; it then takes the form

$$\dot{q} = (D + Eq + Fq^2)q_{xx} + (E + 2Fq)(q_x)^2 - Gq_{xxxx} . \tag{3}$$

We note that the linear terms are the same as in the phase equation, whereas the nonlinear terms change their appearance.

Close inspection of Eq. (2) shows that it can be derived from a Lyapunov functional; that is, we can rewrite Eq. (2) in the form

$$\dot{\psi} = -\delta V/\delta\psi \tag{4}$$

with

$$V(\{\psi\}) = \int dx \left[\frac{D}{2}(\psi_x)^2 + \frac{E}{6}(\psi_x)^3 + \frac{F}{12}(\psi_x)^4 + \frac{G}{2}(\psi_{xx})^2 \right], \tag{5}$$

and where

$$V(\{\psi\}) = \int dx\, U(\{\psi\}).$$

We find that V is strictly nonincreasing in time provided surface terms vanish. This property of $V(\{\psi\})$ guarantees global stability assuming F and G to be positive and demonstrates that $V(\{\psi\})$ is purely diffusive and does not show oscillatory behavior. The quantity $V(\{\psi\})$ guarantees global stability of all solutions of Eq. (2). Since $\dot{V} \leq 0$ the relaxation processes occurring in physical systems associated with Eq. (2) are purely relaxational and do not show oscillatory behavior. We note that such a functional cannot be found straightforwardly and may not exist for the KS equation, since its nonlinearity is not of the gradient type. For the nonlinear phase equation associated with a stationary pattern as is considered here, the transformation $x \rightarrow -x$, $\psi \rightarrow -\psi$ assures that to the order given in Eq. (2) all nonlinearities can be derived from the generalized potential V [Eq. (5)]. Only when higher-order nonlinearities and nonlinearities containing higher-order derivatives are considered is this property lost.

To make close contact to a generalized Ginzburg-Landau energy as is frequently derived to characterize the mean-field behavior close to a phase transition in thermodynamic equilibrium, we rewrite Eq. (5) in terms of the wave vector ($q = \psi_x$):

$$V(\{q\}) = \int dx \left[\frac{D}{2}q^2 + \frac{E}{6}q^3 + \frac{F}{12}q^4 + \frac{G}{2}(q_x)^2 \right]. \tag{6}$$

Analyzing Eq. (6) we note immediately that is has the form of the Ginzburg-Landau energy for a weakly first-order phase transition with the term proportional to G being the analog of the gradient energy.

Pointing out this analogy to equilibrium first-order phase transitions greatly facilitates the interpretation and the analysis of the physical content of Eq. (5). The local wave vector ψ_x acts as the analog of the order parameter and the gradients of the local wave vector characterize the length scale over which the local wave vector changes. Thus the term proportional to G gives us the analog of the coherence length for the changes in the local wavelength. Depending on the signs and magnitudes of D and E in Eq. (5) a number of different scenarios for the nonlinear behavior of a stationary pattern can arise.

First of all we note that for a negative value of the phase diffusion coefficient D the fourth derivative in Eq. (2) must be kept to linearly stabilize the system for large wave vectors. The resulting situation is then similar to that of the KS equation except that the resulting pattern showing a finite-amplitude variation of the wave vector is not irregular. Here we focus on the case of positive phase diffusion coefficient, which is more easily accessible experimentally. This means that a constant wavelength q_0 of the pattern is locally stable centered around $\psi_x = 0$. As pointed out above the cubic nonlinearity is globally stabilizing. To investigate the influence of the quadratic nonlinearity we search for additional stationary solutions of Eq. (2) with constant ψ_x. Since $dV/dt \leq 0$ the system will tend to a state in which $dU/dq = 0$. We find, in addition to $\psi_x = 0$, the two additional roots of $dU/d\psi_x = 0$,

$$(\psi_x)_{2,3} = -\frac{3E}{4F}\left[1 \pm \left(1 - \frac{16DF}{3E^2}\right)^{1/2}\right]. \tag{7}$$

From Eq. (7) and the global shape of Eq. (5) for large q we conclude immediately that $V(\{q\})$ has two local minima and one maximum provided $E^2 > 16DF/3$, that is, $|E|$ must be sufficiently large. From the existence of two local minima it becomes clear that two different wavelengths are locally stable and that one can thus have a stationary pattern where one observes different wavelengths in different parts of the cell. For example, a confined region of larger wavelength can be surrounded by a bulk region of smaller wavelength in a stationary situation. The fourth-order derivative term in Eq. (2) then serves to smoothly connect the two regimes of different wavelengths as a function of space. Inspecting Eq. (2) one notices that D, F, and G can be scaled out by rescaling length, time, and amplitude. That is, aside from the length of the box, Eq. (2) contains only one variable parameter, which we have chosen to be E. In Fig. 1 we have plotted the generalized potential as a function of ψ_x. For small values of the modulus of E, $\psi_x = 0$ is globally stable (i.e., the lowest value of the potential) and there is no other local minimum. As the modulus of E increases, one finds first a nonzero value of ψ_x to be a local minimum and then for even higher values of $|E|$ this local minimum becomes the global minimum. This crossover occurs for $|E| = \sqrt{6}$. At this point it seems important to stress that both the global-stability properties and the values of the wave vectors corresponding to the extrema of the potential are obtained analytically in the case of a stationary pattern as studied here.

More generally we may look for stationary solutions of Eq. (2) by noting that, since $dV/dt \leq 0$, the system will

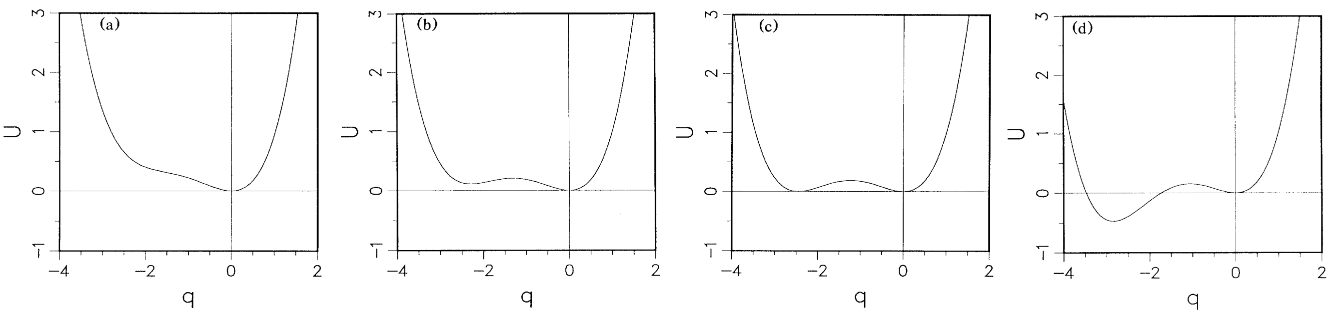

FIG. 1. Plot of the generalized potential as a function of q for $q_{xx}=0$ for different values of E: (a) $E=2.2$, monostable potential; (b) $E=2.4$, local minimum at lower q value appears; (c) $E=\sqrt{6}$, the two minima are equal in depth; (d) $E=2.6$, the second minimum has now become the globally stable one.

tend to a state in which $\delta V=0$, giving

$$Dq+\frac{E}{2}q^2+\frac{F}{3}q^3=Gq_{xx}\,. \tag{8}$$

To demonstrate the importance of the higher-order-gradient term for the coexistence of two different wavelengths in different parts of the cell we have plotted in Fig. 2 a solution of Eq. (8) for $|E|=\sqrt{6}$, the value of E for which the two wells are equal in depth. It is in the vicinity of this value that a solution showing two different values for the wavelength in different parts of the cell will be observed. Going away from this value, the wavelength corresponding to the deeper well for the Lyapunov functional will be preferred to the extent allowed by the constraints imposed at the boundaries.

In closing the discussion on the Lyapunov functional and its properties, we emphasize that the existence of the Lyapunov functional guarantees that q is bounded, regardless of whether the "gradient energy" (proportional to G) is incorporated or not. To guarantee, however, that q_x is bounded the incorporation of the gradient energy is essential for the case where one deals with two wells.

Since the confined states occurring in phase dynamics show a number of similarities to amplitude slugs (localized structures occurring between two stable regimes) that arise in amplitude equations,[19,20] it seems worthwhile at this point to discuss the similarities and differences between the two. Both phenomena are intrinsically nonlinear. Both are subcritical phenomena, meaning that the background state needs to be perturbed beyond a certain level before the state under discussion will form. Both rely on the existence of two basins of attraction (or two wells) so that different regions of space may coexist in different basins. The main difference is that the relaxation process into the wells in phase dynamics occurs diffusively, relying on spatial inhomogeneities for relaxation, whereas the relaxation process in the amplitude equations is of the pure damping type and is independent of spatial inhomogeneities. Therefore, a good term for a confined state in phase dynamics which emphasizes both the similarities and differences is *phase slug*.

Now we come to the comparison with the experiments showing stationary localized states with a wave vector which is different from that in the bulk of the sample. For the case of slot convection studied by Dubois *et al.* (Ref. 9; also see Ref. 21 for a detailed description of the setup), the analogy between the results presented here and the experimental observations is rather striking. For both the experiment and the approach presented here, one finds a region for which the wavelength in part of the cell is different from that in the bulk of the container and both regimes are completely stationary. To further check experimentally the applicability of our approach to the coexistence of states with different wavelengths in slot convection, it would be very important to make abrupt changes in the applied temperature gradients into the regime of Rayleigh numbers for which the localized states have been observed. Our prediction would then be that different lengths for the confined states should become accessible experimentally. Another interesting

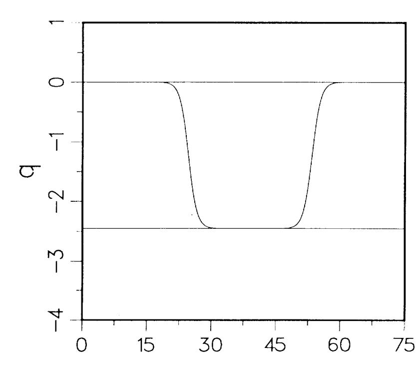

FIG. 2. A stationary solution for $E=\sqrt{6}$; we note the fairly rapid change in wavelength between the two values for the wavelength as is also observed experimentally. The horizontal line at $q=-\sqrt{6}$ corresponds to the local minimum of the potential U given by Eq. (7).

point is that the range over which the stationary confined state exists in experiment is fairly wide, in contrast to the stationary states given by Eq. (8). This may be explained by noting that in experiment the number of rolls in quantized, meaning that the wells must be fairly unequal in depth before the entire system (subject to boundary constraints) will go over into the deeper well. This observation leads to the prediction that longer cells will have a narrower range of parameter values over which stationary confined states exist.

For the case of the dynamic domains observed for the flow between concentric co-rotating cylinders[10,11] the analogy is not quite as immediate as for the case of slot convection, since for the Taylor instability the wavelength variation is not strictly one dimensional, but has spatial and temporal variations in the azimuthal direction as well. Focusing on the direction parallel to the cylinder axis, however, the same global picture as that outlined above for slot convection emerges. One has two locally stable states existing in different parts of the gap along the cylinders. To capture the azimuthal motion as well it will then be necessary to incorporate also a phase equation describing the azimuthal pattern. This can be done along the lines discussed in Refs. 7 and 8 for the Taylor wavy mode and for the modulated Taylor-wavy-mode state. From such an analysis it emerges (Ref. 7) that the propagative phases in the azimuthal direction, which obey equations of KS type, couple back to the phase describing the location at the vortices in the axial direction. It is this cross coupling which can render the motion irregular in time in the axial direction as well.

For the case of a spiral pattern one obtains due to the lack of mirror symmetry both types of nonlinearity characteristic of phase dynamics,[13] namely the KS nonlinearity associated with propagative phases and the gradient-type nonlinearity discussed in detail in this paper as it is associated with phases describing a stationary pattern. In this case the nonlinear equation reads

$$\dot{\phi} - v\phi_x - C\phi_{xxx} = D\phi_{xx} + G\phi_{xxxx} + E\phi_x\phi_{xx} + F(\phi_x)^2\phi_{xx} + H(\phi_x)^2 . \qquad (9)$$

We note that no Lyapunov functional is known for this equation and we speculate that it might show the coexistence of a stationary roll pattern and of a spatially and temporally irregular state, since it contains both types of nonlinearities.

In conclusion, we have shown how one can obtain in the framework of phase dynamics, stationary states for which different wavelengths coexist in different parts of an experiment cell. We have demonstrated that this is an intrinsically nonlinear phenomenon. For a phase variable associated with a stationary pattern a Lyapunov functional is obtained. We have critically analyzed recent experiments on slot convection and the Taylor instability for co-rotating cylinders and we find qualitative agreement with the results presented here. In addition, we have suggested experiments to further test the concept presented.

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