Scaling Theory of Self-Organized Criticality

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We study the phenomena of self-organized criticality originally proposed by Bak, Tang, and Wiesenfeld. A continuous-energy model is introduced. Using numerical simulations, we find that energy is homogeneously and isotropically distributed in space, and that it is concentrated around discrete values. We propose a scaling theory to estimate the various exponents. The activation-cluster size distribution is found to be $D(s)\sim 1/s^{\tau}$, $\tau=2-2/d$; and the dispersion relation $t\sim r^{z}$, $z=(d+2)/3$.

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Bak, Tang, and Wiesenfeld[1] (BTW) have recently introduced a sandpile model to describe the so-called self-organized critical (SOC) phenomena. A remarkable feature of their model is the ever-amplifying, self-adjusting activation processes at all length and time scales. The BTW model has many possible applications in various realms of science; e.g., it models[2] earthquake mechanisms. It is perhaps the simplest model that captures the characteristics of a vast class of spatial and temporal evolution processes. The BTW model has since been studied in the fashion of traditional critical phenomena and extensive numerical simulations have been carried out.[3] A host of scaling exponents and their relations have been proposed and numerically measured.[3] Subsequently, Obukhov[4] developed an ϵ-expansion renormalization-group scheme, which enabled him to predict the upper critical dimension $d_c=4$, along with the one-loop calculation of the exponents.

In this Letter we analyze SOC processes as a transport problem of a conserved quantity, which for convenience we call *energy* (this terminology was also used by BTW[1] on some occasions). We work with a variant of the BTW model in which the energy variable assumes *continuous* values. We believe[5] this choice should not alter the universality scaling class. Our numerical simulations are intended to explore special features of this continuous-energy model. A salient result is that in critical states, energy is concentrated around a few "quantized" values. Using energy conservation and the local nature of energy transfer in our model, we derive the distribution law of the size s of activation clusters: $D(s)\sim 1/s^{\tau}$, $\tau=2-2/d$ for $\infty > d \geq 1$, which agrees with the results obtained in special cases (for $d=2$ and 3) of previous simulations.[1,3] Using an argument of local repulsion among the activation events, we find that the relaxation time t of an activation cluster of linear size r obeys the dispersion relation $t\sim r^{z}$, $z=(d+2)/3$ for $d\leq 4$, which is also in reasonable agreement with the numerical findings.[3]

Description of the model.—Let us take a hypercubic d-dimensional lattice; energy E can be stored on each node. At time t, an input energy δ $(0<\delta<1)$ is added to that of a randomly chosen node at $\mathbf{r}$, such that $E(\mathbf{r},t+1)=E(\mathbf{r},t)+\delta$. $E(\mathbf{r},t)$ thus assume non-negative continuous values, and this process repeats itself. There is a limit, $E_{\max}$, on the allowed energy on any node. If, at a given time t, the energy on the node $\mathbf{r}$ is $E(\mathbf{r},t)>E_{\max}$, an activation event will occur with the following consequence: The full amount of energy $E(\mathbf{r},t)$ will be transferred, in equal parts, to its $2d$ nearest neighbors and it is reset to zero. The transferred energy, in its turn, acts as input energy for the neighboring nodes. Energy on these nodes increases and, whenever the limit $E_{\max}$ is again reached, further activation, energy transfers, ensues. Without loss of generality, we take $E_{\max}=1$ throughout this work. A single input-energy event may trigger off activation on a set of connected nodes; we call this set an activation cluster, or avalanche.[1] The boundaries are assumed isotropic[6] and, in the thermodynamic limit, at infinity. We adopt free boundary conditions: $E(\text{boundaries},t)=0$. By energy conservation, the transferred energy is eventually let out through the boundaries.

We assume that energy input takes place at a rate so slow that the processes are *adiabatic*: The system has to be allowed to quiet down (relax) before input energy is again introduced. This adiabatic condition is intended to avoid possible complications of cluster-cluster interaction. Thus the system only responds (activation may or may not happen) to the introduction of input energy, otherwise it is quiescent. After a sufficiently long time the average value of the stored energy reaches a plateau, which we call the critical energy storage E_c. The system is then said to be in dynamical equilibrium, or critical. If the average stored energy is too high, the system becomes particularly prone to activation, and energy is more rapidly transferred out. The opposite will happen if the average stored energy is below the critical E_c. The two conflicting attempts, to maximize the total energy content by the introduction of input energy and to stabilize the system by relaxing and transferring energy, compromise to sustain equilibrium.

We have numerically measured the critical energy: for $d=2$, $E_c=0.62\pm 0.01$, and for $d=3$, $E_c=0.57$

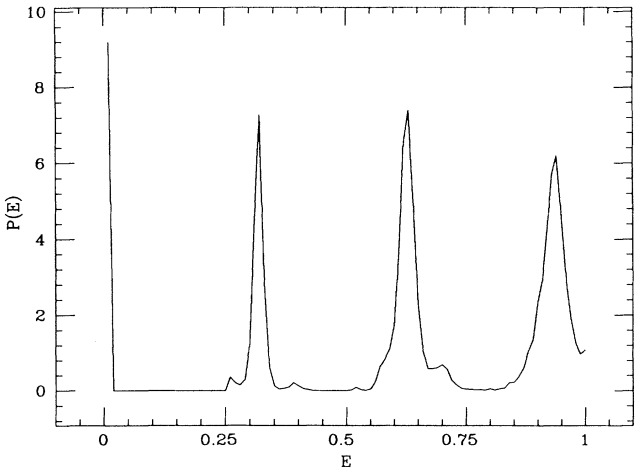

FIG. 1. The energy distribution of a single node $P(E)$ in critical states, for a 60×60 square grid. It is obtained by averaging over all the nodes of the system and over a long time interval. Input energy was chosen continuously between 0 and δ $(=\frac{1}{2})$ to avoid possible artificial influence by the input amount. All energy values can be approximately packed into four peak values. In ascending order, they represent roughly 10%, 16%, 32%, and 32% of the total nodes, respectively. The nodes of the last three values are colored by green, blue, and red in Fig. 2.

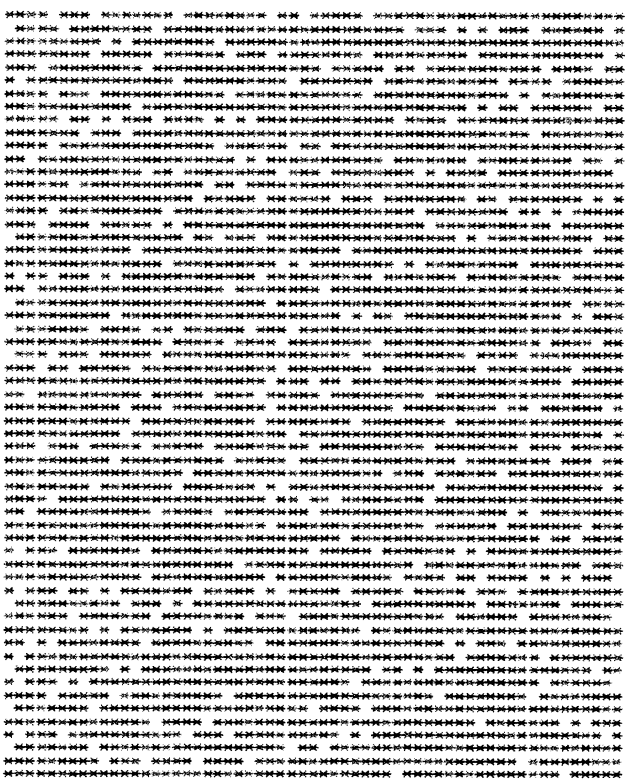

FIG. 2. A snapshot of the energy distribution on a 60×60 grid. The four peak values in Fig. 1, in ascending order, are represented by blanks, and green, blue, and red asterisks, respectively. The red nodes are most likely to "fire": They will be activated by a small provocation (energy from input or neighbors). The blue ones can be considered "weak nodes" in conducting activation: Two or more firing neighbors are needed to "ignite" one such node. The green ones and blanks are weaker still. There are not as many blanks as one would expect—this is because when a node has fired, it may receive "backfiring" from its neighbors, so it rarely empties its content. Thus it is hard to identify avalanches that have just occurred and the potential ones from this picture. To illustrate the isotropy and homogeneity of the critical energy distribution, we have purposely introduced input energy only on half of the grid—it is the reader's guess which half.

±0.01.[7] (As for BTW, in our version of the model $d=1$ is also a trivial case, where $E_c\sim1$ and the whole system is a single cluster.) Figure 1 shows a typical energy distribution of the dynamical equilibrium state of a two-dimensional system. Energy is concentrated around four distinct peaks. Except for the peak at $E=0$, the other three peaks have finite spreads. In general, there are $2d$ peaks for the hypercubic lattice with the nearest-neighbor interaction. This shows that energy can be transferred only in multiples of a finite *quasi*unit. These peaks represent the probability of occupation in *quasi*-units, and may be interpreted as dynamical attractors. It is remarkable that energy is "quantized" in such a way in critical states.[8] We believe the finite spreads are due to intrinsic dynamical fluctuations.

What is more remarkable is that we have observed in our simulations that the deposited energy $E(\mathbf{r},t)$ is homogeneously distributed all the way to the boundaries, independently of where input energy is introduced. Suppose we drop more energy near the origin and less near the boundaries. The system, nevertheless, organizes itself to be *homogeneous* and *isotropic,* on the average. Homogeneity and isotropy are achieved by activating nodes more or less frequently, responding to possible inhomogeneous input energy. A color rendering of a critical state is given in Fig. 2.

In the following we derive the distribution law of activation-cluster sizes. We ask the following question: If we drop an input-energy unit at a given node, say at the origin and at time $t=0$, what is the expected energy (it does not necessarily include any of the input energy) that has flowed into a node at a distance r away from the input as the consequence? Denote by $S(\mathbf{r},t)=\langle E(\mathbf{0},0)\times E(\mathbf{r},t)\rangle$ this correlated, or transferred energy, where $\langle\cdots\rangle$ is the sample average, over activation clusters of all sizes. Suppose that the system is already critical; the input energy then has to be dispelled to the boundaries via activation. Since energy transfer in our model is by a local isotropic mechanism, and since the transferred energy $S(\mathbf{r},t)$ is averaged over activation clusters of all sizes, it must satisfy the relaxation equation, in a contin-

uum notation:

$$dS(\mathbf{r},t)/dt = \Delta S(\mathbf{r},t) + \delta^d(\mathbf{r})\delta(t) ,$$

where Δ is the Laplacian. (This is to be compared with the discussion below for the evolution of a *single* cluster.)

The average total new energy received at the node $\mathbf{r}$, due to the input event, is proportional to $T(\mathbf{r}) = \int_0^\infty dt\, S(\mathbf{r},t)$. It satisfies the static equation: $\Delta T(\mathbf{r}) = -\delta^d(\mathbf{r})$. Traditionally, $\mathbf{j} = -\nabla T$ is called the energy current; we can express the energy conservation law as $\int \mathbf{j}\cdot d\boldsymbol{\sigma} = 1$, where $d\boldsymbol{\sigma}$ is the $(d-1)$-dimensional surface element. At large r we have the solution $T \sim 1/r^{(d-2)}$, from which we obtain the outgoing, radial component of the energy current $j_r = -\partial T/\partial r \sim 1/r^{(d-1)}$.

Denote by $D(r)$ the distribution function of activation clusters of the *linear* size r. Energy can be transferred to the node $\mathbf{r}$ only through activation clusters that contain *both* the input node and the node $\mathbf{r}$. Therefore, all the activation clusters of linear size equal to or greater than r contribute to the received energy $T(\mathbf{r})$. We obtain the relation $T(\mathbf{r}) = \int_r^\infty dr' D(r')$, where we have assumed that each activation cluster contributes equally to $T(\mathbf{r})$. In other words, each activation cluster *locally* transfers an equal amount of energy per unit volume, no matter how large its size ($>r$). We have $\partial T/\partial r = -D(r)$, or from the above discussion we obtain the activation-cluster distribution function $D(r) \sim 1/r^{(d-1)}$. Note that for $d=1$, we have $D(r) \sim 1$, so clusters of *infinite* linear size can occur with a probability of 1.

We assume that the activation clusters are *compact* objects with rough boundaries, more like the Eden[9] clusters than *fractals*. This assumption is supported by direct inspection of simulation patterns, and by the previously measured fractal dimensions[1,3] which are very close to the compact dimension d. When an activation cluster is advancing, it is unlikely to leave large regions behind unaffected, in contrast to diffusion-limited-aggregation fractal clusters where long-range forces are present. In Fig. 2 one expects that activation clusters will most likely develop along connected red nodes. However, the "weak nodes," such as the blue or green ones, are also able to conduct activation when surrounded by other "firing" nodes. There are, of course, small patches (candidates are blanks, green, and even blue nodes) left inactivated—but this is unlikely to give rise to fractals. This is consistent with our previous assumption that activation clusters transfer local energy homogeneously.

Denote now by s the size of an activation cluster; using the relations $s = r^d$ (compactness) and $D(s) = D(r)dr/ds$, the above distribution law can be rewritten as $D(s) \sim 1/s^\tau$, $\tau = 2(1-1/d)$. This exponent τ can be readily compared to the reported numerical results,[3] $\tau \sim 1$ for $d=2$, and 1.33 for $d=3$. Our analytical prediction is in good agreement with these numerical results.

Next we study how a single activation cluster evolves in time. Around the average homogeneous energy E_c there are regional fluctuations. The reason for a single activation cluster to occur is that excess energy, when perturbed by an input energy, must dissipate. In general we expect that larger clusters need more time to grow. This expressed by the dispersion relation[3] $t \sim r^z$, where r is the linear size of the cluster and z is the dynamical exponent. We can regard an activation cluster as the trace of t consecutive *activation fronts* when they *sweep* over a region of linear size r. If these activation fronts were free to evolve and isotropically diffuse through space, we should expect the diffusion behavior $t \sim r^2$. There is, however, an effective repulsive interaction[10] among these t activation events: Nodes that are activated previously by the very same activation cluster are less likely to be activated again, since they are just relaxed. Though backward propagation is allowed and observed in our simulations, the activation fronts prefer the "fresh" nodes.[11] This repulsive interaction among the t consecutive activation events drives the cluster to expand outward, faster than would be implied by diffusion alone.

Denote by $\rho = t/r^d$ the density of the activation fronts within the volume $V = r^d$. Since the repulsive interaction is *local,* we postulate that the effective expansion potential is $U = V\rho^2$, in analogy with polymer studies.[12] The outward velocity dr/dt should be equated with the driving force $-\partial U/\partial r$.[13] The above relation leads to $r \sim t^\nu$, $\nu = 3/(d+2)$, or $z = 1/\nu = (d+2)/3$. This result is also in reasonable agreement with BTW's measurements (for $d=2$, $z=1.29$; for $d=3$, $z=1.70$). The above prediction indicates that the upper critical dimension is $d_c = 4$. Above d_c the local repulsive interaction has negligible effect, since the expansion rate cannot be smaller than that of diffusion; i.e., for $d \geq 4$ we have $z=2$.

It may seem that the activation fronts are a kind of branching true self-avoiding walk (TSAW).[14] However, there is an important qualitative difference: In the TSAW the repulsive potential is $V\rho$ rather than $V\rho^2$. This is because the TSAW has only a single active site, the walker, which can terminate when crossing the previously visited sites, whereas the activation fronts in our model can branch indefinitely. The above discussion is not sensitive to what shape and size the activation fronts may have.

In conclusion, we have analyzed a continuous-energy model, which is a variant of the BTW model of SOC phenomena. We find that energy is concentrated around a few discrete values. In dynamical equilibrium states there is a well-defined critical stored energy and it is isotropically, as well as homogeneously, distributed in space, even when energy input is not.

Critical phenomena have one characteristic in common: correlation lengths diverge at criticality. It is instructive to highlight how (diverging) long-range corre-

lations arise in our model. Energy transfer, after being averaged over avalanches of all sizes, is via diffusion.[3] Energy received at a large distance r away from an input node decays following the power law $T(r) \sim 1/r^{(d-2)}$. It would be a rather trivial scaling, or long-range correlation, if energy transfer were by smooth, continuous diffusion. The curious feature of our model is that activation on a distant node is only *occasional*: either it happens, when homogeneous energy per volume will be transferred, or *not*. A large avalanche is bound to happen, as required by conservation, to supply energy on a distant node, not matter how small the amount. The above power law by diffusion dictates how often they should occur—this leads to the distribution function $D(s)$.

We use the fact that when a single activation cluster is evolving in time, the previously activated nodes are less likely to be activated again, to deduce a local repulsive interaction for the t consecutive activation events. Note that the above dispersion relation is the same as that for a self-repelling chain,[12] the reason being that in both cases there is the same local repulsive interaction, despite their apparently different geometries. The analytical predictions for the above two exponents (other exponents are related to these two,[3] plus the compactness assumption) agree reasonably well with the results of previous simulations.[1,3]

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[1]P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987); Phys. Rev. A **38**, 364 (1988).

[2]P. Bak and C. Tang (to be published).

[3]C. Tang and P. Bak, Phys. Rev. Lett. **60**, 2347 (1988).

[4]S. P. Obukhov, in *Random Fluctuations and Pattern Growth,* edited by H. E. Stanley and N. Ostrowsky, Cargese Proceedings 1988 (Kluwer, Dorcrecht, 1988); (to be published).

[5]In fact, we have verified the scaling of the activation-cluster distribution and obtained the exponent τ very close to BTW's results for $d=2,3$, though not with as good statistics as theirs.

[6]In the original BTW sandpile model, anisotropic boundaries were sometimes used. We believe different boundaries should not affect the criticality and scaling relations, as long as energy (or the sandpile "slope") is conserved.

[7]Using a mean-field argument, L. Pietronero (private communication) has obtained analytical estimates which are very close to these values.

[8]It appears that in their "prescient" paper (Ref. 1) BTW have chosen just the "right" values for the sandpile building rule: for the $d=2$ square lattice they worked with four allowed values $E=0,1,2,3$ which correspond to our four peaks, upon normalization.

[9]M. Kardar, G. Parisi, and Y.-C. Zhang, Phys. Rev. Lett. **56**, 889 (1986).

[10]Similar repulsive interaction was first noticed by Obukhov (Ref. 4) in his ϵ-expansion work.

[11]Note that we do not consider interactions with *previous* activation clusters. In other words, an evolving activation cluster interacts only with itself, not with the background. When the system is quiescent it is as *critical* as ever.

[12]P.-G. de Gennes, *Scaling Concepts in Polymer Physics* (Cornell Univ. Press, Ithaca, London, 1979).

[13]This type of argument (balancing velocity to force) was first proposed by L. Pietronero, Phys. Rev. B **27**, 5887 (1983), for the true self-avoiding walks.

[14]D. J. Amit, G. Parisi, and L. Peliti, Phys. Rev. B **27**, 1635 (1983).

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