Spontaneous Transition from Flat to Cylindrical Solitons

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(Received 16 May 1989)

Flat, cylindrical, and spherical soliton solutions to various model equations are known. Many of these exact solutions have been seen in numerical simulations. However, there are few simulations that actually show that exact flat solitons can break up into an array of exact cylindrical or spherical solitons and follow this on a step by step basis. This Letter presents the first of these two kinds of transitions for the Zakharov-Kuznetsov equation governing ion acoustic solitons in strongly magnetized plasmas.

PACS numbers: 47.20.Ky, 52.35.Kt, 52.35.Py, 52.35.Sb

In a three-dimensional unmagnetized plasma, small-amplitude, flat ion acoustic solitons are stable.[1-3] The existence of these stable entities has been confirmed experimentally.[4]

The situation changes if a strong external magnetic field is applied to the plasma. Although the basic solutions are still the same for a soliton propagating along the magnetic field **B**, the dynamics in the perpendicular direction will be altered, causing solitons to lose their stability. Planar solitons become unstable if the field is strong enough and the perpendicular wavelength of the perturbation exceeds some critical value. A good model for this situation is furnished by the Zakharov-Kuznetsov (ZK) equation.[5-9]

The Zakharov-Kuznetsov equation for ion acoustic waves propagating along the magnetic field in a strongly magnetized, two-component cold plasma is[8,9]

$$n_t + nn_x + (\Delta n)_x = 0\,, \qquad (1)$$

$$\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2\,, \qquad (2)$$

where the independent variables x,y,z,t are obtained from the laboratory ones x_l, y_l, z_l, t_l by appropriate stretching which involves the external magnetic field in the case of y and z ($x=\epsilon[x_l-t_l]$, $y \propto \epsilon y_l$, $z \propto \epsilon z_l$, $t=\epsilon^3 t_l$). Here n is the normalized deviation of the ion density from the average.

Although flat ion acoustic soliton solutions of the Zakharov-Kuznetsov equation are unstable, spherical solutions exist and are more robust. The original paper[8] includes a limited stability analysis of these entities.

The above mentioned work on flat soliton stability is based on a linear analysis describing the onset of instability. The main goal of this Letter is to investigate the further fate of unstable plane solitons. Because of numerical limitations, we consider the two-space-dimensional version of the ZK equation, but return briefly to three-dimensional considerations at the end of this paper. Thus we now assume invariance in z.

A particular class of solution of (1) is given by the following function of x,t:

$$n = 12\eta^2 \operatorname{sech}^2[\eta(x-x_0-4\eta^2 t)]\,. \qquad (3)$$

One of these one-dimensional, one-soliton solutions will be essentially the starting point of our numerical solution. We take (3), perturbed periodically in the y direction. The results are presented in Fig. 1. Cylindrical solitons are seen to result after a while. Further evolution, not shown here, leaves these entities unaltered. All this would suggest that we have obtained a new, stationary and, in fact, stable solution of the ZK equation. Analytic candidates for this solution can be obtained from ZK if we consider solutions to (1) in the form $n(x,y,t)=f(x-ct,y)$ that vanish at infinity. We obtain

$$\Delta n - (c-n/2)n = 0\,. \qquad (4)$$

This equation is a two-dimensional version of a full equation cited in the original ZK paper.[8] Figure 1(e) suggests we look for angle-independent solutions to (4); thus

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial n}{\partial r}\right) - \left(c-\frac{n}{2}\right)n = 0\,. \qquad (5)$$

This is an ordinary differential equation and is easily solved. The resulting shapes and velocities for the soliton have been checked to be in full agreement with those of the cylindrical solitons emerging in Fig. 1. Thus the transition from a flat to an array of cylindrical solitons, all of which are found from theory, has been demonstrated step by step on a computer. This is the most important point of our Letter. Its significance may well have bearing on other fields of physics.

These new structures emerged as a results of the instability of a flat soliton and, of course, we must now consider *their* stability. In the following we show that they are two-dimensionally stable with respect to dilations. Consider scalings that conserve momentum $P=2\pi\int n^2 r\,dr$ (in this model normalized n and v are equal, so the integrand could be written in the more familiar form nv in place of n^2):

$$n \to \lambda^{-1} n\,, \quad r \to \lambda r\,.$$

The "energy" is now

$$E(\lambda r, \lambda^{-1} n) = -I_1/3\lambda + I_2/\lambda^2\,,$$

$$I_1 = 2\pi\int n^3 r\,dr\,, \quad I_2 = 2\pi\int n_r^2 r\,dr\,,$$

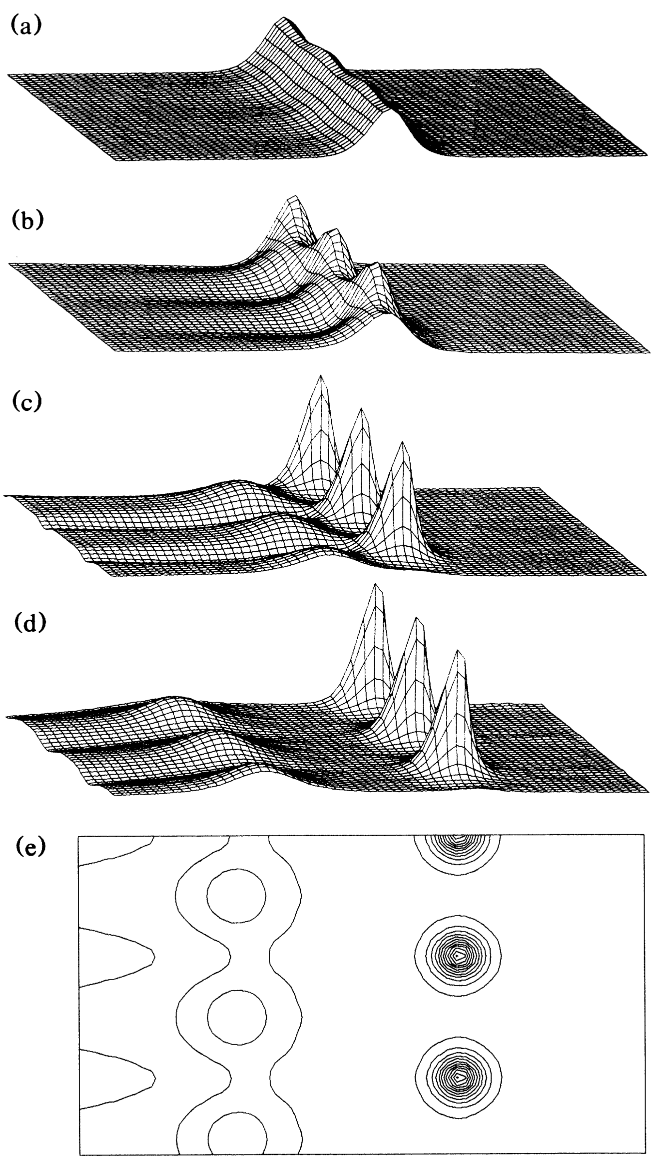

FIG. 1. (a)-(d) Consecutive stages in transition from flat soliton to cylindrical array and (e) bird's eye view of the last stage corresponding to (d). The coordinate system is moving with the velocity of the initial soliton.

and so

$$\partial E/\partial\lambda = I_1/3\lambda^2 - 2I_2/\lambda^3 .$$

Thus, $\partial E/\partial\lambda = 0$ (and this is a minimum) for $\lambda = 1$ if $I_1/I_2 = 6$. This was easily verified to be the case for our solution and I_1/I_2 is invariant for all soliton solutions of (5) (the value 5.999 was obtained; we also used our solutions to check that the total energy of the cylindrical array is lower than that of the initial flat soliton).

We now propose to summarize the results from a three-dimensional point of view. Flat solitons are unstable if the system permits long enough perturbations. If the perturbation is also z independent the flat soliton eventually breaks up into a system of cylindrical solitons. By analogy we again expect a fully three-dimensional analysis to show a cylindrical soliton breaking up into a system of spherical solitons *when long enough perturbations along z are possible.* The spherical solitons are presumed stable. If indeed there is also a critical wavelength for perturbations that can destroy cylindrical solitons, instabilities will not always set in and these tube-shaped entities might be observed experimentally in some situations.

Two other open questions are what happens to flat solitons if the initial perturbation is three dimensional, and what are the mechanics of possible further transitions from tube-shaped to spherical solitons (growth rates of instabilities, etc.).

In conclusion, we see that an unstable perturbation can lead to a spontaneous transition from a flat to a cylindrical soliton. We also see that the mechanics of this transition from one soliton structure to another, both known analytically, can be followed. However, new questions have been raised and these will be investigated in a fuller version of this paper, where more details of the emerging cylindrical solitons will also be given.

The authors would like to thank T. Czerwińska, P. Goldstein, M. Młodzianowski, and H. A. Rose for stimulating discussions.

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