Ghosts and the c Theorem

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We discuss the central charge in continuous families of two-dimensional conformal field theories. In a nonunitary conformal field theory the c theorem does not hold: A critical line can have a continuously varying central charge. This is illustrated with the bosonic-string ghost system and may pose difficulties for attempts to formulate string theory on the space of 2D field theories.

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Recent progress in two-dimensional conformal field theory (CFT) has led to important applications in string theory and in the study of critical phenomena. Most attention has focused on unitary theories, in part because this assumption is a powerful constraint and leads toward a classification, and in part because the spacetime background of string propagation may be represented by unitary CFT.[1-3]

In this Letter we address the behavior of both unitary and nonunitary theories which possess a critical line of renormalization-group (RG) fixed points. The reparametrization ghosts of string theory are the most important example of a nonunitary theory. Indeed, when the allowed backgrounds consist of all $c=26$ CFT's, many of which have no spacetime interpretation at all, it is only the ghosts which make a string a string.

We reexamine carefully the conditions for the existence of a critical line and show how the central charge may be computed perturbatively, completing arguments of Zamolodchikov[4] and Cardy[5] for a line generated by a conserved current. Zamolodchikov's "c theorem," which implies that the central charge is constant along such a line, may fail to hold if Ward identities or operator products for a marginal operator are anomalous. This means that for a theory with nonunitary degrees of freedom, such as string theory, conformal invariance is not sufficient to guarantee stationarity of the central charge in the space of 2D theories. Of course, the c theorem was formulated for unitary theories, so there is no contradiction, but this may pose problems in describing string dynamics on this space. We illustrate the issues with a detailed analysis of the free-boson model and of the bosonic-string ghost system. We conclude with remarks on the impact for string theory.

Zamolodchikov's remarkable c theorem[4] states that when a theory with action S_0 is perturbed, $S_0 \to S_0+S'$, where $S'=\sum \kappa_i \int d^2w\, V_i(w,\bar{w})$, there is a function c on the space of 2D theories which is stationary with respect to κ_i if and only if S_0 is conformally invariant. If S_0 is conformally invariant, then c is equal to the central charge.[6]

Unitarity is essential for the first part of the theorem. The theory S_0 requires a short-distance cutoff. By differentiating correlation functions of stress-energy tensor components, Zamolodchikov derives via a renormalization-group equation for the cutoff theory that $\beta^i \partial c/\partial\kappa_i = -12\beta^j\beta^k G_{jk}$, where $G_{jk}=|z-z'|^4 \times\langle\Phi_j\Phi_k\rangle|_{z=z'}$ for fields Φ_j appearing in the expansion of the trace of the stress tensor. Unitarity implies positive definiteness of G_{jk}, and it follows that the condition $\partial_i c=0$ implies $\beta^j=0$ and the theory is conformally invariant.[4,7]

The converse is more elusive. If $\beta^j=0$ and $T^\mu_\mu=0$, we learn nothing from this relation since both sides vanish. A different analysis is needed to show that $\beta=0$ implies $\partial_i c=0$. In the case where there is a critical line of RG fixed points, Cardy[5] states that not only is c stationary, but $c=1$ (or the theory is "decomposable" into theories with $c=1$) along a line if there are no conserved spin-2 currents besides T and the marginal operator has no mixing with other operators.

We proceed by computing the central charge c directly from the two-point function of the stress-energy tensor. Below, and above, we take a Euclidean signature world sheet with flat metric and we use complex coordinates $w=w_1+iw_2$.

In a conformally invariant, RG fixed-point CFT, the value of the central charge can be obtained[2] from $\langle T(z)T(z')\rangle = c/2(z-z')^4$.

A perturbation has two consequences for $\langle TT\rangle$: S' changes the measure with respect to which the correlation function is taken, and S' may give an additional piece to the stress tensor, $T'_{\alpha\beta}=(-4\pi/\sqrt{g})\delta S'/\delta g^{\alpha\beta}$.

We can calculate $\langle TT\rangle_{S_0+S'}$ perturbatively around the theory defined by S_0:

$$\langle (T_0+T')(T_0+T')e^{-S'}\rangle_{S_0} = \frac{c(\kappa)/2}{(z-z')^4}\,.$$

Expanding to first order in κ,

$$\langle T_0T'\rangle_{S_0} + \langle T'T_0\rangle_{S_0} - \langle T_0T_0S'\rangle_{S_0} = \frac{\Delta c/2}{(z-z')^4}\,. \qquad (1)$$

This must vanish for c to be stationary.

We now consider a single, marginal operator[5,8,9]

$V(w,\bar{w})$ with dimensionless coupling constant κ, generating a one-parameter family of CFT's, $S(\kappa)=S_0+S'$. For S' to give an invariant contribution to the action, V must have scaling dimension (1,1) for all values of κ. Therefore the operator-product expansion of $V(z,\bar{z})\times V(z',\bar{z}')$ cannot contain V. If there is a term $VV \sim C|z-z'|^{-2}V(z',\bar{z}')$, the perturbation changes its own conformal weight[8] and is "marginal" only for $\kappa=0$:

$$\langle VV\rangle_{S_0+S'} \approx |z-z'|^{-4}(1+\kappa C\ln|z-z'|^2) \approx |z-z'|^{-4+2\kappa C}.$$

On the other hand, it is *not* necessary, as has sometimes been stated, to require that $\langle VV(S')^n\rangle_{S_0}=0$ for all n. It could be that these higher functions are merely proportional to the free result: $\langle VV(S')^n\rangle_{S_0} \propto \langle VV\rangle_{S_0}$. Then the critical exponent remains unchanged in $\langle VV\rangle_{S_0+S'}$, and the operator remains marginal. This is illustrated below.

We consider now the important case of a marginal perturbation where the marginal operator is a product of conserved currents (which may be regulated by point splitting and normal ordered), $S'=(\kappa/2\pi)\int d^2w\, j_w j_{\bar{w}}$. The induced correction to the stress tensor is $T'=(\kappa/2){:}j_w j_w{:}$. Now $j(w)$ should be a weight-(1,0) current with the operator-product expansion[2]

$$T(z)j(w) \sim \frac{j(w)}{(z-w)^2}+\frac{\partial_w j(w)}{z-w} \sim \frac{j(z)}{(z-w)^2}. \quad (2)$$

If we evaluate $\langle T_0T_0V\rangle$, and integrate, $\langle T_0T_0S'\rangle = \langle T_0T'\rangle + \langle T'T_0\rangle$, and then comparing with (1) shows that $\Delta c=0$ to first order. This naive expectation is not valid if (2) fails to hold. We check the validity of this argument in two CFT's with marginal perturbations: first, the unitary Gaussian model, or free boson on a circle, and second, the nonunitary string ghost system.

Let us first show, perturbatively, that rescaling a field or changing the compactification radius of a free boson has no effect on the central charge. The free-boson action is $S_0=(1/2\pi)\int d^2z\, \partial_z\phi\, \partial_{\bar{z}}\phi$, with stress tensor $T_0=-\frac{1}{2}{:}\partial_z\phi\,\partial_z\phi{:}$.

Observe that there is a weight-(1,0) current $j_w=i\partial_w\phi$ and a weight-(0,1) current $j_{\bar{w}}=-i\partial_{\bar{w}}\phi$, which combine to form the marginal operator $V=\partial_w\phi\,\partial_{\bar{w}}\phi$. We need only the two-point functions of the currents, obtained from $\langle\phi(z,\bar{z})\phi(z',\bar{z}')\rangle=-\ln|z-z'|^2$:

$$\langle j_z j_{z'}\rangle=\frac{1}{(z-z')^2},\quad \langle j_{\bar{z}} j_{z'}\rangle=\pi\delta^{(2)}(z-z'). \quad (3)$$

Evaluating $\langle T_0T_0\rangle_{S_0}$ at finite separation gives $c=1$.

Perturbing the theory, $S'=\kappa S_0$ changes the normalization, $S_0\rightarrow(1+\kappa)S_0$. The stress tensor picks up an additional term $T'=-(\kappa/2){:}\partial_z\phi\,\partial_z\phi{:}=\kappa T_0$.

When the target space is a circle, there are primary operators $\exp[\pm i\phi(z)]$ and we identify $\phi\cong\phi+2\pi$; i.e., ϕ lives on a circle of unit radius. The effect of S' can be absorbed by rescaling the fields, $\phi\rightarrow\phi(1+\kappa)^{1/2}$, and changing the compactification radius. (In this way one can show exactly that $c=1$ for any radius, but we proceed perturbatively.)

Because $\langle T_0T'\rangle=\kappa\langle T_0T_0\rangle\neq0$, the individual contributions to Δc in (1) cannot separately vanish. Fortunately, $\langle T_0T_0S'\rangle\neq0$:

$$\frac{\kappa}{8\pi}\int d^2w\langle{:}j_zj_z{:}{:}j_{z'}j_{z'}{:}{:}j_wj_{\bar{w}}{:}\rangle=\frac{\kappa}{(z-z')^4}.$$

The contributions to (1) cancel, so to first order $\Delta c=0$, as expected. It is essential to retain the current-conserving contact term of (3) whenever we have antiholomorphic fields that will be integrated over the entire surface.[10]

Does c remain constant to higher orders? Although the main interest is the first correction, it is instructive and amusing to compute the higher-order corrections:

$$\langle T_0T_0(S')^n\rangle_{S_0}=(n+1)!\kappa^n\frac{1/2}{(z-z')^4}.$$

Several types of contractions contribute to the connected part of the correlation function. Each contraction gives an identical contribution after the integrations are performed, using the formula $\int d^2w(w-z)^{-2}(\bar{w}-\bar{z}')^{-2}=\pi^2\delta(z-z')$, which can be checked by integration by parts. This yields the perturbation expansion for $\langle T_0T_0e^{-S'}\rangle$ to all orders, and it may be resummed for $|\kappa|<1$:

$$(1-2\kappa+3\kappa^2\cdots)\frac{1/2}{(z-z')^4}=\frac{1}{(1+\kappa)^2}\langle T_0T_0\rangle_{S_0}.$$

The full two-point function is then given exactly as

$$\langle TT\rangle_{S_0+S'}=(1+\kappa)^2\langle T_0T_0e^{-S'}\rangle_{S_0}=\langle T_0T_0\rangle_{S_0}.$$

Therefore $c(\kappa)=c(0)=1$ to all orders, the well-known result that a free boson on a circle has a critical line of $c=1$ theories. It was essential in perturbation theory that the correlators with $(S')^n$ be nonzero and that contact terms be retained.

A similar calculation finds that $\langle VVe^{-S'}\rangle=(1+\kappa)^{-2}\times\langle VV\rangle$, so that the fields may be rescaled but the critical exponent of the marginal operator is indeed unchanged. It is this relaxed criterion that is appropriate for marginal operators.[11]

Next we consider a marginal perturbation to the nonunitary CFT of the string reparametrization ghosts. The theory is described by a b-c system[12] of free anticommuting fields of spin λ and $1-\lambda$, with action $S_0=(1/\pi)\int d^2w(b\,\partial_{\bar{w}}c+\bar{b}\,\partial_w\bar{c})$. The central charge is $c=1-3(2\lambda-1)^2$. For $\lambda=\frac{1}{2}$, the theory is unitary and is equivalent to the free boson considered above with $c=1$. For $\lambda=2$, the theory is nonunitary and describes the negative-norm ghosts of the bosonic string with $c=-26$.

There is a weight-(1,0) fermion-number current j_w

$=:bc:$, classically conserved on a flat world sheet, which has an anomalous operator product[12] for $\lambda\neq\frac{1}{2}$:

$$T_0(z)j(w)\sim\frac{2\lambda-1}{(z-w)^3}+\frac{j(z)}{(z-w)^2}+\text{finite}. \qquad (4)$$

For definiteness, we set $\lambda=2$. From this current, a marginal perturbation may be constructed, $V=j_w j_{\bar{w}}=bc\bar{b}\bar{c}$. This is just the conformally invariant Thirring interaction. This theory is defined for $\kappa>-1$ and has been thoroughly studied in fermionic and bosonic form by Freedman *et al.*[13] For simplicity we use the bosonic language and carry over our previous calculations.

We begin with the bosonic action corresponding to the free fermionic theory,[12]

$$S_0=\frac{1}{2\pi}\int d^2w\,\partial_w\phi\,\partial_{\bar{w}}\phi+\frac{3i}{4\pi}\int d^2w\sqrt{g}R\phi\,,$$

$$T_0+\hat{T}_0\equiv-\frac{1}{2}:\partial_z\phi\,\partial_z\phi:-\frac{3i}{2}\partial_z^2\phi\,,$$

and perturb by adding $V=j_w j_{\bar{w}}=\partial_w\phi\,\partial_{\bar{w}}\phi$, where j_w is now the bosonic current and V is the same operator that appeared for the free boson. It is interesting that the perturbation is proportional to a piece of the free action in the bosonic formulation but not in the fermionic.[14] [This current $(j_w,j_{\bar{w}})$ is conserved, $\partial_{\bar{w}}j_w+\partial_w j_{\bar{w}}=0$, and is dual to the anomalous fermion-number current[12,13] $(-j_w,j_{\bar{w}})$.] The induced change in the stress tensor is $T'=\kappa T_0$, as before. The only change from the free-boson case is the "improvement term" in the stress tensor, $\hat{T}_0=-\frac{3}{2}\,\partial_z j_z$.

Expanding the two-point function $\langle TT\rangle_{S_0+S'}$,

$$\langle (T_0+T')(T_0+T')e^{-S'}\rangle_{S_0}+\langle \hat{T}_0\hat{T}_0 e^{-S'}\rangle_{S_0}\,,$$

the first term is the free-boson result, giving $c=1$. We concentrate on the second term. The zeroth-order contribution gives $c=-27$. The first-order contribution from $\langle\hat{T}_0\hat{T}_0S'\rangle$ gives $\Delta c=27\kappa$, and $\partial c/\partial\kappa\neq0$. Solving to all orders, we find $\langle\hat{T}_0\hat{T}_0(S')^n\rangle=-27n!\kappa^n/2(z-z')^4$. Hence $c(\kappa)=1-27(1-\kappa+\kappa^2\cdots)$, which we resum for $|\kappa|<1$ to give the final result

$$c(\kappa)=1-\frac{27}{1+\kappa}=-26+\frac{27\kappa}{1+\kappa}\,, \qquad (5)$$

in agreement with Ref. 13, where $c(\kappa)$ was computed from the trace T^μ_μ by functional-integral and index-theorem techniques.

Note that $c(\kappa)$ is monotonically increasing for the allowed range $\kappa>1$ [continuing (5)], from $c(-1)=-\infty$ to $c(\infty)=+1$. $c(\kappa)$ has no stationary point, even for the free theory $\kappa=0$. Although $c\geq0$ for $\kappa\geq26$, the theory is never unitary.[15]

We have taken a first step in studying the space of nonunitary 2D quantum field theories. A vanishing β function is not sufficient to ensure that the central charge is stationary on the space of 2D theories, and the c theorem does not hold. The conformal Ward identities[2] do not suffice because a perturbation can be *marginal* without being *primary*. Cardy's result[5] that $c=1$ in a unitary theory with a marginal operator required that the operator have no mixing with other fields and that there be no other spin-2 operators in the operator-product expansions. However when there is a current, $\partial_w j_w$ can and generally does mix. Such currents can be the very reason that the marginal operator even exists, and we have seen how to account for their mixing.

All we used, in fact, is the stress tensor and the current structure of the perturbation. The obstacle to deriving results for arbitrary $V(w,\bar{w})$ is that without some knowledge of the structure it is difficult to treat the contact terms mixing w and $\bar{w}$ and to relate V to T'. For relevant perturbations, the latter problem is absent. For the supersymmetric ghost action with a Thirring interaction, $c(\kappa)$ is linear,[16] and hence nowhere stationary. We have not studied this theory, but presumably higher-order contributions to $c(\kappa)$ vanish by supersymmetry.

Coupling this system to a "matter" CFT with $c=26$ yields a combined $c=0$ unitary string theory, and one can ask what happens to the coupled system when the ghosts are given a marginal or a relevant perturbation. How does this drive changes in the matter? (The matter system itself, to describe physical Minkowski spacetime, also contains a nonunitary degree of freedom for the timelike polarization of the string.)

These considerations may be relevant for attempts to formulate string theory on the space of all 2D field theories.[7,17] In this approach, motivated by the many appealing features of string perturbation theory, the conformally invariant theories are identified as the "classical solutions" to the "equations of motion," and the c function plays a special role as a candidate action whose stationary points correspond to these perturbative ground states. Critical lines represent connected degenerate vacua. In this language it is possible to begin formulating questions about tunneling and nonperturbative effects. If we take the proposal seriously, we should include the ghost degrees of freedom. But the classical solutions will be unstable against perturbations of the ghost degrees of freedom, and some other principle (e.g., Becchi-Rouet-Stora-Tyutin invariance) besides vanishing total central charge may be needed to stabilize the theory.

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Note added.—After this work was completed, we learned of an interesting paper by Elitzur, Giveon, and Rabinovici[18] in which a different deformation of the free massless b-c ghost system is discussed. These authors consider a mass term, a relevant operator which breaks conformal invariance and generates a RG flow to a $c=0$ theory. Hence they consider the family of free, massive theories flowing from a critical point, whereas above we considered the family of interacting, massless conformally invariant theories along a critical line of fixed points. In Ref. 18, conditions are given where stationary c does not imply conformal invariance, and above we have given conditions where conformal invariance does not imply stationary c. In both of these cases, it is possible for the central charge of the ghost theory to increase. It would be desirable to understand both types of perturbations together as part of the set of all perturbations of a CFT.

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[1]For recent reviews of 2D CFT, see the lectures of P. Ginsparg, in "Fields, Strings, and Critical Phenomena," edited by E. Brézin and J. Zinn-Justin (Elsevier, New York, to be published), and of J. Cardy, *ibid.*

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