Hierarchical Models for Aging and Relaxation of Spin Glasses

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We show that the aging phenomena found in spin glasses and other complex systems can be reproduced by a hierarchical model of relaxation.

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Aging effects were first observed in spin-glass systems[1-7] by the Uppsala group, and have quite recently been measured in high-T_c superconductors[8] and charge-density-wave systems[9] as well. They are one important experimental indication of a relaxation process proceeding in a complex phase space, with many local energy minima and a broad spectrum of relaxation times. This paper discusses a master-equation approach to the problem for temperatures below the spin-glass transition temperature. In this range the model is capable of reproducing the most important features of the experiments with only two fitting parameters. For the zero-field–cooled (ZFC) experiments the features are a kink in the magnetization plotted as a function of logarithmic time at $t=t_w$ and a corresponding maximum of its derivative as shown in the insets in Figs. 2 and 3. The underlying model assumption is that the stochastic dynamics of any system with a complex state space can—for flow temperatures—be coarse grained into a random walk on a tree.[10-13] The same kind of approach was previously applied to the calculation of the ac linear susceptibility of spin glasses,[14] and the theoretical arguments supporting this view are described in a recent paper by the authors.[10]

Other recent theoretical approaches to the same problem are due to Koper and Hilhorst[15] and to Fisher and Huse.[16] Our work is different from those in several respects. Firstly, it is a hierarchical rather than a parallel model of relaxation. Secondly, the scaling properties of the model are not put in the theory of an *Ansatz*, but are rather derived from a simple master equation with exponential waiting times. Thirdly, it is a dynamical model[17] as the nonequilibrium properties for spin glasses and other systems[8] are only weakly related to equilibrium properties on the time scales which are exponentially accessible at very low temperatures.

In our model the low-energy states of the system are organized in a tree structure as is shown in Fig. 1. The nodes of the tree are obtained by lumping together sets of states of the spin-glass system. These states are close to each other in the sense that the system can move from one to the other with a small energy change. Each local minimum node in the tree represents thus a phase-space pocket in which the system can be trapped. To move from one local minimum to the other, the system has to cross energy barriers by performing short-range hops between neighbor nodes. Possible paths are given by the connectivity of the tree. This differs from ultrametric models with long-range hops between the states at the base line of the tree,[18,19] except at temperatures close to zero. The clustering of the states in our model as well as in Ref. 18 is by construction only a matter of time scales. In our model there is no claim of ultrametricity with respect to the spin overlap function which is so far left unspecified. For a thorough discussion of the construction and the transition to a regular tree see Ref. 10.

Within the linear-response approach used below we need only to specify the disorder average and the disorder correlation function of the coarse-grained magnetization of the different nodes. Since we only aim at a low-temperature theory, the physical properties of the energetically higher-lying nodes will only be of exponentially small importance, and for any disorder realization we set their magnetization to zero. The magnetization of the bottom nodes will be of the order of the square root of the size of the system. Disorder averaging these quantities in addition, we set them to zero. Thus the only quantity left is the disorder correlation function between two bottom nodes which is assumed to decay linearly

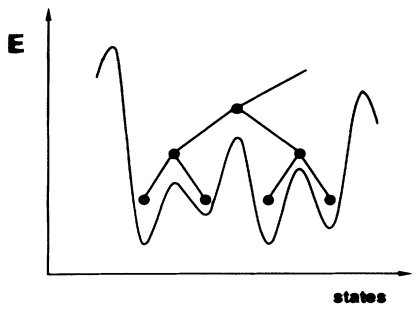

FIG. 1. The coarse-grained structure of the phase space of a spin glass. The nodes of the tree represent a large number of states of the system which can be reached from one another by spin flips with a small limited amount of energy.

with the distance along the base line of the tree. This assumption is quite arbitrary; however, it has already been used successfully in Ref. 14. Its physical implication is that equal changes in the disorder correlation function become harder and harder to achieve as the relaxation progresses. This effect requires that the distance defined by the disorder correlation function be different from the ultrametric distance defined by the relaxation times.

Let the time at which the system was quenched be $-t_w$. During the time t_w the system relaxes unperturbed, and at time zero a small field is turned on. The initial condition of the second part of the relaxation thus depends on t_w. In order to get an idea of how it can produce an aging effect, we can think of the probability distribution after time t_w as being uniform on the bottom nodes of some subtree (of characteristic size proportional to logarithm of the time) and zero otherwise. Then for times t less than t_w the diffusion process is acting on a uniform distribution with zero net effect. For t of the order of t_w the system begins to diffuse appreciably outside the initial subtree and the relaxation effect becomes noticeable. For t much larger than t_w the effect of the initial distribution wanes away, and one has the usual algebraic relaxation. This qualitative picture of the relaxation process is supported by the following quantitative analysis.

According to linear-response theory the magnetization $M(t)$ at time t due to a small field $H(t)$ is given by

$$M(t)=\frac{1}{k_BT}\int_{-\infty}^{t}H(t')R(t,t')dt', \tag{1}$$

where T is the temperature and k_B the Boltzmann constant. In the usual equilibrium situation[20] one then relates the memory function R to the autocorrelation function of the stochastic dynamics by the well-known fluctuation dissipation theorem,

$$R(t,t')=\frac{d}{dt'}\langle M(t)M(t')\rangle . \tag{2}$$

Our situation is different since we are dealing with a nonequilibrium situation. This gives rise to two additional terms in Eq. (2). The first term is a phase-space average of the magnetization. As the disorder average of the magnetization is zero for all nodes this term vanishes. The second term can be considered as a "response to the initial condition." It is a phase-space average of a function which vanishes identically in equilibrium due to detailed balance. A thorough discussion which will be published separately shows that within the Kramers approximation[10] used below this term is of no importance.

In the ZFC experiment the system is prepared by quenching without field at time $-t_w$ into a certain state, which we take in our model to be state zero. At time 0 a small constant field H is switched on and we find from Eq. (1)

$$M(t)\frac{k_BT}{H}=\langle M(t)M(t)\rangle-\langle M(t)M(0)\rangle . \tag{3}$$

The systems thermal relaxation is described by a Metropolis-like random walk on the tree of Fig. 1. Let the branching ratio of the tree be z. The upward transition rate from one node to its parent node is taken as $\kappa e^{-\Delta/T}$ while the downward rate is set equal to 1. The parameter κ is a measure of the size of the phase-space region which every node represents and Δ is the energy difference between a node and its parent. The linear increase of the free energy with the level height is an *a priori* arbitrary model choice. However, it guarantees the applicability of the Kramers approximation[10] used below, and is *a posteriori* supported by our results. Other choices of the energy dependence might nonetheless be worth exploring.

Let $P(l,t\,|\,i)$ be the probability that the system is at the bottom node l at time t, given that it was at the bottom node i at time zero. Here the bottom nodes are numbered in ascending order starting with zero from left to right.

Then the random-walk autocorrelation function is given by

$$\langle M(t)M(0)\rangle=\sum_{l=0}^{n}\sum_{i=0}^{n}\left(1-\frac{2}{n}\,|\,l-i\,|\right)\times P(i,t_w\,|\,0)P(l,t\,|\,i),$$

where t is the time spent under the influence of the field, and n is the total number of low-energy states. Note that $\langle M(t)M(0)\rangle$ is not invariant under time translations. As $\langle M(t)M(t)\rangle=1$ due to $P(l,0\,|\,i)=\delta_{l,i}$ and due to Eq. (3), we then have

$$\frac{M}{H}=\frac{2}{nk_BT}\sum_{l=0}^{n}\sum_{i=0}^{n}|\,l-i\,|\,P(i,t_w\,|\,0)P(l,t\,|\,i). \tag{4}$$

We now use the Kramers approximation[10] as a step towards a closed-form analytical formula for the model ZFC magnetization. The physical meaning of the approximation is that in any one subtree the system relaxes to a quasiequilibrium situation on a time scale short compared to the time needed for a substantial loss of probability out of the subtree. That means that the probability distribution is the same as in equilibrium, but modulated by an exponentially decaying time-dependent envelope. This envelope function, which is the probability of being in a subtree of height m at time t, is then

$$Q^m(t)=\exp(-\lambda_m t),$$

where λ_m is the rate of escape from the mth subtree. λ_m is assumed proportional to the equilibrium probability of being at the "top" of the subtree: $\lambda_m\sim A^m$ with $A=(\kappa/z)e^{-\Delta/T}$. We assume $A<1$, which means a restriction on the possible values of the temperature if $\kappa>z$. The mathematical basis of the Kramers approximation and its limitations are detailed in Ref. 10.

The diffusion propagator $P(l,t\,|\,i)$ depends only on the ultrametric distance between its spatial arguments, i.e.,

the height k of the smallest subtree connecting l and i, and within the Kramers approximation it can be written as

$$P(z^k,t)=\sum_{m=k}[Q^m(t)-Q^{m-1}(t)]z^{-m}.$$

With this notation the leading terms for the magnetization from Eq. (4) are

$$\begin{aligned}M\frac{k_BT}{H}=&d_f(0,t)P(0,t_w)+P(1,t_w)P(1,t)\frac{z(z-1)}{2}\\&+\frac{1}{2}\frac{z-1}{z}\sum_{m=1}^{N-1}P(z^{m-1},t_w)z^m\sum_{i=m+1}^{N}[Q^i(t)-Q^{i-1}(t)]z^i+\frac{1}{2}\frac{z-1}{z^2}(1+z^3)\sum_{m=1}^{N-1}P(z^m,t)P(z^m,t_w)z^{3m}\\&+\frac{1}{2}\frac{z-1}{z^2}\sum_{m=1}^{N-1}[Q^m(t_w)-Q^{m-1}(t_w)]z^{2m}P(z^m,t)-\frac{1}{2}\frac{z^2-1}{z^2}\sum_{m=1}^{N-1}[Q^N(t)-Q^m(t)]z^{2m}P(z^{m-1},t_w).\end{aligned}\tag{5}$$

This formula can be evaluated numerically with limited computational effort for reasonably large trees. However, it is also convenient to have a closed-form, albeit (more) approximate, expression for the time-dependent magnetization.

The needed additional mathematical approximation retains the only important feature of the function $Q^m(t)$, i.e., its decay on a certain time scale λ_m^{-1}. We approximate $Q^m(t)$ by

$$Q^m(t)=\begin{cases}1-\lambda_m t, & 0\le t\le\lambda_m^{-1},\\ 0, & \text{otherwise.}\end{cases}$$

With $\gamma=-\ln z/\ln A$ and taking in addition the limit $N\to\infty$ we find from Eq. (5), as leading terms in t and t_w,

$$M(t,t_w)\frac{k_BT}{H}=\alpha_1\left\{\alpha_2\left[\frac{t_<^2}{t_>}\right]^\gamma+\alpha_3\left[\frac{t_<}{t_>}\right]t_>^\gamma+\left[(t^\gamma-t_w^\gamma)+\alpha_4\left(\frac{t_w}{t}t^\gamma-\frac{t_w^{2\gamma}}{t^\gamma}\right)\right]\theta(t-t_w)\right\},\tag{6}$$

with $t_<=\min(t,t_w)$, $t_>=\max(t,t_w)$, and

$$\alpha_1=\frac{1}{2}\frac{(1-A)^3z}{(1-Az)},\quad \alpha_2=\frac{[(z^4+z^3-z+1)A-2z^2](A-1)}{(z^2+z+1)(A-z)(Az^2-1)},$$

$$\alpha_3=\frac{A[(z^2-2z-1)Az+(2z-1)A^2z-(z-1)A^3z+1](z-1)}{(A^2z-1)(Az^2-1)(A-z)},\quad \alpha_4=\frac{(A-1)(z-1)^2A}{(Az^2-1)(A-z)}.$$

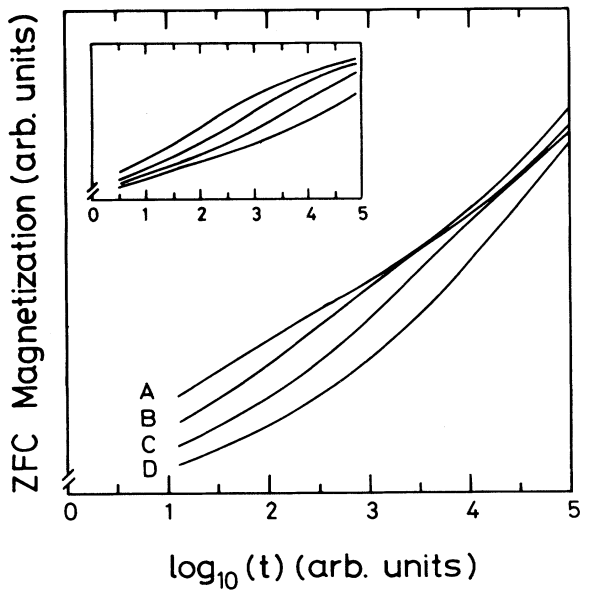

FIG. 2. The ZFC magnetization as a function of the logarithm of time for four different waiting times t_w. Curve A, $t_w=10^2$; curve B, $t_w=10^3$; curve C, $t_w=10^4$; curve D, $t_w=10^5$. Inset: Experimental data over the same time range, taken from Ref. 3.

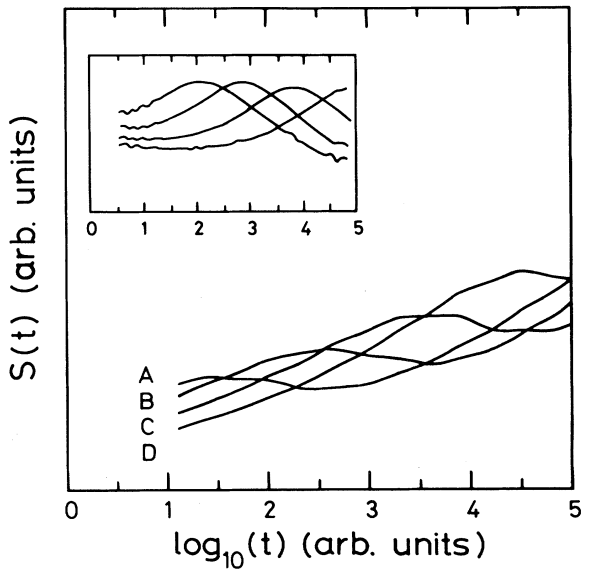

FIG. 3. The logarithmic derivative $S(t,t_w)=\partial M(t,t_w)/\partial\ln t$ as a function of the logarithm of time for four different waiting times t_w. Curve A, $t_w=10^2$; curve B, $t_w=10^3$; curve C, $t_w=10^4$; curve D, $t_w=10^5$. Inset: Corresponding experimental data taken from Ref. 3. Note that both sets of curves have maxima roughly at $t=t_w$.

We have checked that the numerically evaluated results and the above analytical formula agree very well.

The following results for the time dependence of the ZFC magnetization $M(t,t_w)$ and its derivative $S(t,t_w)=\partial M(t,t_w)/\partial \ln t$ were computed according to Eq. (5) after a rescaling of time and numerical differentiation for various parameter values of κ, z, and T and for different tree sizes N. The plots in Figs. 2 and 3 were all obtained for $z=1.15$, $\kappa=2$, and $T=0.5$.

Figure 2 displays the values of the model ZFC magnetization as a function of logarithmic time, for different values of the waiting time, while Fig. 3 shows the corresponding values of the logarithmic derivatives. The insets show the corresponding experimental data from Ref. 3.

As seen from the drawing, as well as from the analytical formula Eq. (6), the numerical result $M(t,t_w)$ increases for $t>t_w$ as a power law of t with a small exponent and "something happens" on a logarithmic scale for $t\approx t_w$. This made explicit by looking at the logarithmic derivative $S(t,t_w)$ of $M(t,t_w)$ in Fig. 3.

By comparing with the experimental results of Svedlindh *et al.*,[3] we find that for temperatures *low* relative to the energy spacing of the tree, the characteristic features of the experiments, i.e., the slow algebraic dependence of the ZFC magnetization on both t and t_w and the maximum in its logarithmic-time derivative at $t=t_w$, are well reproduced by the model, provided that the model parameters z and κ are close to unity.

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