Interaction of Localized Solutions for Subcritical Bifurcations

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We discuss the interaction of localized solutions as they arise for the subcritical bifurcation to traveling waves. We find that for a large parameter range the localized solutions can interact so that they emerge after the collision with a size and shape unchanged compared to that well before the collision. The mechanism for this behavior, which is unusual for a strongly dissipative system, is qualitatively different from that associated with solitons for completely integrable systems. In accord with this we find that for other parameter values counterpropagating localized solutions can annihilate.

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One of the big breakthroughs in nonlinear physics in the last 25 years was the discovery of solitons by Zabusky and Kruska.[1] They found that for the Korteweg–de Vries equation, as it arises, for example, for shallow-water waves, localized solutions could collide and emerge after the collision unchanged in speed, size, and shape, when compared to the time well before the collision. Since then the field of solitons has seen tremendous development and it has become clear that completely integrable equations such as the Korteweg–de Vries equation, the sine-Gordon equation, and the nonlinear Schrödinger equation can be characterized by an infinite number of conservation laws.[2–4] Aside from the particlelike one-soliton solutions, which have been known for over 100 years, these equations have also been found to have multisoliton solutions and their collision behavior has also been studied (see, e.g., Ref. 5 for a detailed review).

All the equations which are known to be exactly integrable are of the Hamiltonian or purely dispersive type. That is, dissipation is discarded entirely or at best taken into account perturbatively in the limit of very weak dissipation. On the other hand, for many macroscopic phenomena in physics dissipation is not just a small perturbation, but plays an important role in the determination of the dynamic behavior. Therefore it seems natural to ask whether it is possible to have a strongly dissipative system whose solutions share at least some of the properties with those prototype systems known to be exactly integrable; whether it is, for example, possible to have particlelike solutions of an equation of evolution, which collide and interpenetrate, but which are unchanged in speed, size, and shape after the collision.

The purpose of this Letter is to give an example for such a behavior for a prototype equation, which is *both* strongly *dissipative* and *dispersive*. The equation of interest arises as the envelope equation for a weakly subcritical bifurcation to counterpropagating waves.[6–10] For such a bifurcation the amplitude does not grow continuously from zero as a function of a control parameter, but has a small jump and shows hysteretic behavior as this control parameter is reduced from a value above threshold. The envelope equation is derived using the vicinity to onset of the instability and a sufficiently slow variation in space and time for the wave packet (cf. Refs. 11 and 12 for reviews on envelope equations). For such an equation, which arises naturally for a number of physical systems including the onset of oscillatory convection in binary fluid mixtures,[6,7] we find for a large range of parameter values for the nonlinear interaction between counterpropagating waves, localized solutions which emerge unchanged after collision, but which are due to a mechanism quite different from that in solitonic systems. In addition, this equation also shows interesting interaction behavior, including complete interpenetration as well as partial annihilation, for collisions between localized solutions corresponding to a single-particle and to a "two-particle" state. The limitations of the approach presented are critically examined and the need for more analytic work is emphasized.

To derive the envelope equation for weakly subcritical bifurcations to traveling waves one keeps all the terms which are needed to lowest consistent order in the distance from the onset of the instability. For waves traveling in one direction this equation reads for slow spatial modulations in one spatial direction,[6,7]

$$\partial_t A + v\,\partial_x A = \chi A + (\gamma_r + i\gamma_i)A_{xx} - (\beta_r + i\beta_i)\,|A|^2 A - (\delta_r + i\delta_i)\,|A|^4 A - (\lambda_r + i\lambda_i)\,|A|^2 A_x - (\mu_r + i\mu_i)A^2 A_x^* . \quad (1)$$

The real part of the cubic term, β_r, is negative for an inverted bifurcation, whereas the real part of the quintic term $\delta_r > 0$, and thus provides the saturation for the amplitude A and γ_i,β_i and δ_i are associated with spatial and nonlinear dispersion, respectively. v is the linear group velocity of the waves. As has been discussed first in Refs. 6 and 7 the nonlinear gradient terms ($\sim\lambda,\mu$) arise to the same order in the distance from the onset of the instability and must therefore be kept. It can be easily checked that three of the parameters in Eq. (1) can be scaled out by rescaling length, time, and amplitude. In addition, one can go into the moving frame if an unbounded or periodic motion in only one direction is studied. This leaves, however, still eight parameters. This situation must be contrasted with the one obtained for the prototype completely integrable systems [sine-Gordon, Korteweg–de Vries (KdV), nonlinear Schrödinger], which typically do not have any free parameters.

It has been shown[13] that Eq. (1) allows for spatially localized, particlelike solutions over a large range of parameter values; and it has been argued[13] that the stability of these localized structures is essentially due to a nonvariational effect, namely a feedback mechanism between frequency and amplitude of the resulting solution.

In Fig. 1(a) we show two such particlelike solutions, neglecting the nonlinear gradient terms in Eq. (1). These localized solutions change their shape in the presence of these terms. As will be discussed in detail elsewhere,[14] these nonlinear gradient terms can lead, in addition, to a slowing down of the localized solutions, an observation which could be relevant in connection with experimental observations in binary fluid convection.[15] The numerical technique is described in Ref. 14.

Starting from many different initial conditions, including Gaussians with variable amplitude and width, typically either one or several, well separated, one-particle states are obtained, which all have the same shape (width and height). For a finite subclass of initial conditions, however, it is possible to obtain in the long-time limit a state which we will call a "two-particle" state. A typical example is plotted in Fig. 3(a) on the left. This object is stable against small perturbations. It is also formed if two one-particle states are brought sufficiently close together. Its width as well as the depth and the width of the dip are fixed and always the same for a given set of parameters in Eq. (1). If one prepares initially two one-particle states closer together than this characteristic distance between the two peaks of the two-particle state, they move apart, the dip forms, and a two-particle state results. If one prepares the two one-particle states too closely together, however, the object collapses and forms a single one-particle state. The two-particle state, whose behavior under collisions with one-particle states will be discussed further below, thus constitutes a fairly robust object, and also exists over a large range of parameters in Eq. (1), which seems to be a subrange of the existence of one-particle states. We would like to note, however, that it takes, from a given initial condition, a much longer period of time (about 2 orders of magnitude) to form a two-particle state than to form a one-particle state. If the initial Gaussian is not large and wide enough, it collapses to zero, and no one- or two-particle state is formed. The same applies to parameter values outside the range for the existence of one- and two-particle states for all initial conditions as long as the state with zero amplitude is locally stable.

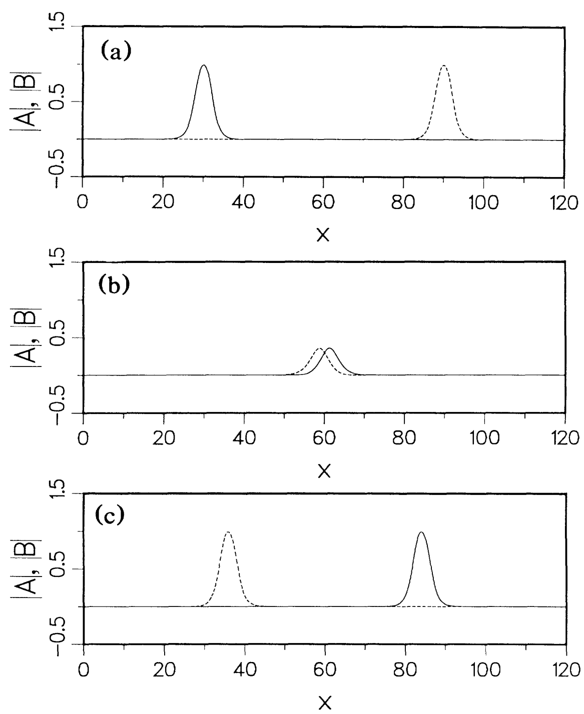

FIG. 1. Complete interpenetration of two one-particle solutions ($\xi_r = 2.5$): (a) initial approach, (b) interaction, and (c) final state after interaction.

To study collision between counterpropagating one- and two-particle states, we start from the coupled envelope equations for counterpropagating waves, which read, generalizing Eq. (1) in the usual way,

$$\partial_t A + v\,\partial_x A = \chi A + (\gamma_r + i\gamma_i) A_{xx} - (\beta_r + i\beta_i)|A|^2 A - (\delta_r + i\delta_i)|A|^4 A - (\xi_r + i\xi_i)|B|^2 A - (\lambda_r + i\lambda_i)|A|^2 A_x$$
$$- (\mu_r + i\mu_i) A^2 A_x^* - (\tilde{\lambda}_r + i\tilde{\lambda}_i)|B|^2 A_x - (\tilde{\mu}_r + i\tilde{\mu}_i) B^* A B_x - (\zeta_r + i\zeta_i) A B B_x^* \quad (2)$$

and a corresponding equation for the left-traveling wave. In writing down these equations, we have discarded quintic, spatially homogeneous cross-coupling terms (like those $\sim |B|^2|A|^2 A, |B|^4 A$) in Eq. (2).

To study collisions between localized states, all one needs to do is to start, for example, with two initial Gaussians sufficiently far apart so that they have none or only a negligibly small interaction. Then they will quickly form one-particle states, whose interaction can then be studied. In Fig. 1 we have plotted the temporal evolution for a stabilizing cubic cross coupling ($\xi_r = 2.5$) between the counterpropagating waves.[16] Comparing Figs. 1(a) and 1(c), we see that the shapes of the one-particle solutions in the asymptotic regimes before and after the collisions are identical, whereas during the interaction process itself a compound object is formed, which has a reduced amplitude when compared to that of the one-particle solution [Fig. 1(b)]. The latter aspect is similar to that obtained for the KdV equation in the pioneering paper of the field.[1] That the amplitude during the interaction is reduced in the present case can be intuitive-

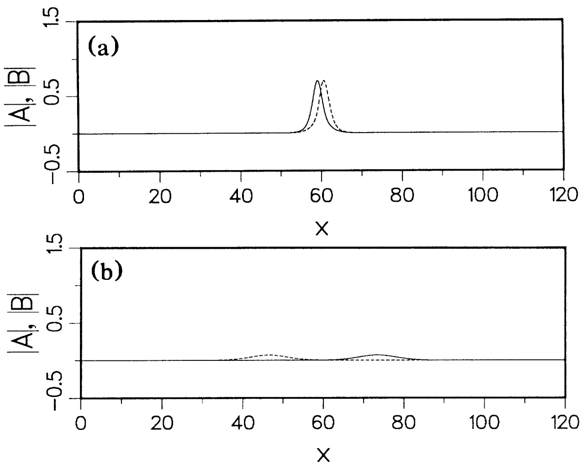

FIG. 2. Annihilation of two one-particle solutions; the initial state is identical to Fig. 1(a) ($\xi_r = 3$): (a) interaction and (b) decaying remnants of the two one-particle solutions.

teraction is reduced in the present case can be intuitively understood by keeping in mind that the interaction parameter ξ_r is stabilizing and thus tends to reduce the amplitude of the particle propagating in the opposite direction. A growing value of ξ_r leads to a stronger suppression of the amplitudes during the interaction process. As long as the size assumed during the interaction is larger than a critical limit, the one-particle solutions grow quickly and assume their universal shape after the interaction is finished. If the size drops during the interaction below this threshold value due to a further increase in ξ_r, however, the two one-particle states annihilate each other completely, that is, nothing is left over in the long-time limit, since the remnants surviving the collision decay to zero. This is shown in Figs. 2(a) and 2(b); the initial state is the same as in Fig. 1(a) except for the change in ξ_r ($\xi_r = 3$). The phenomenon of annihilation clearly demonstrates that Eq. (2) is not exactly integrable in general; one would indeed not have expected this to hold for a strongly dissipative system. The observation, however, that one-particle solutions can collide and interpenetrate with unchanged shapes clearly demonstrates that such a behavior can also arise in strongly dissipative systems for a range of parameter values.

To explore further the similarities and the differences with the dispersive, exactly integrable systems, we have investigated the collision between two- and one-particle states, again as a function of ξ_r keeping all the other parameters fixed. In Fig. 3 we have plotted the time sequence for $\xi_r = 1$. Similar to Fig. 1, it is possible that one has complete interpenetration, even if a one- and a two-particle state interact. Figures 3(a) and 3(c) show the asymptotic regimes well before and well after the collisions, whereas Fig. 3(b) shows a snapshot taken during the interaction process. This interaction is rather complex. Similar to one-particle collisions one has generally a reduction in amplitude. In addition, it seems that the one-particle state interacts first predominantly

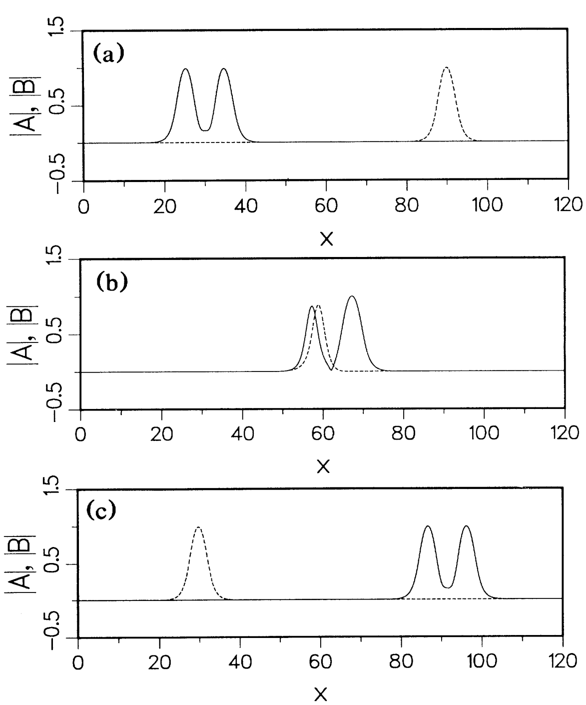

FIG. 3. Interpenetration of a one- and a two-particle state ($\xi_r = 1$): (a) initial configuration, (b) interaction, and (c) final state.

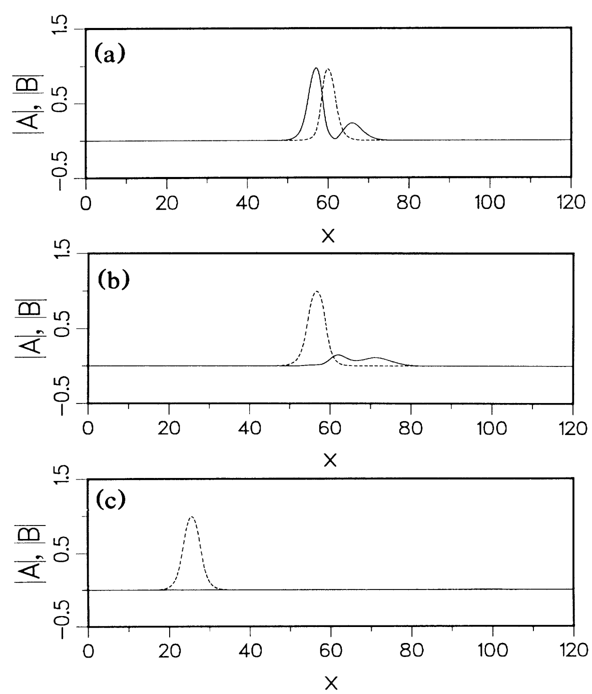

FIG. 4. Annihilation of a two-particle solution by a one-particle state; the initial state is as in Fig. 3(a) ($\xi_r = 2$): (a) interaction starts, (b) interaction nearly finished, and (c) final state.

with the first half of the two-particle state leaving the second half nearly unchanged. Then the interaction with the second half takes place: It might be worth noting that the remnant of the one-particle state is somewhat reduced in area during this latter part of the interaction.

In Fig. 4 we show an effect which we have not observed for collisions between well formed one-particle states. For $\xi_r = 2$ the one-particle state can annihilate the two-particle state completely, but is itself left completely unscathed in the asymptotic regime after the collision. Close inspection shows that half of the two-particle state is slightly narrower than the one-particle state. This explains why the one-particle state survives the collision.

Aside from the complete interpenetration and the annihilation, we have also observed other scenarios for the interaction between one- and two-particle states. To conserve space we discuss just one more example. In Fig. 5 we show that it is possible that during the interaction the second half of the two-particle state is quenched ($\xi_r = 1.5$). That is, a collision between a two- and a one-particle state can lead as a result of the interaction to two one-particle states, a phenomenon also unknown from completely integrable systems. We close this description by noting that all these processes have been observed over a range in parameter space. We have not been able, however, to achieve complete annihilation of a one- and a two-particle state thus far.

In this Letter we have studied particlelike solutions of the envelope equations for counterpropagating waves as they arise close to a weakly inverted bifurcation. For this prototype equation we have shown that one- and two-particle solutions can interact and interpenetrate completely with their final size and shape well after the collision being identical to that well before the interaction, a feature obtained for solitons in exactly integrable systems with an infinite number of conservation laws. In the present strongly dissipative system, however, we find for different parameter ranges also completely different types of behavior including complete annihilation of two one-particle states and the annihilation of a two-particle state when it collides with a one-particle state. Since the nonlinear evolution equation studied here arises in real physical systems, is of prototype character, and shows a wide range of unusual phenomena, further analytic results would be highly desirable.

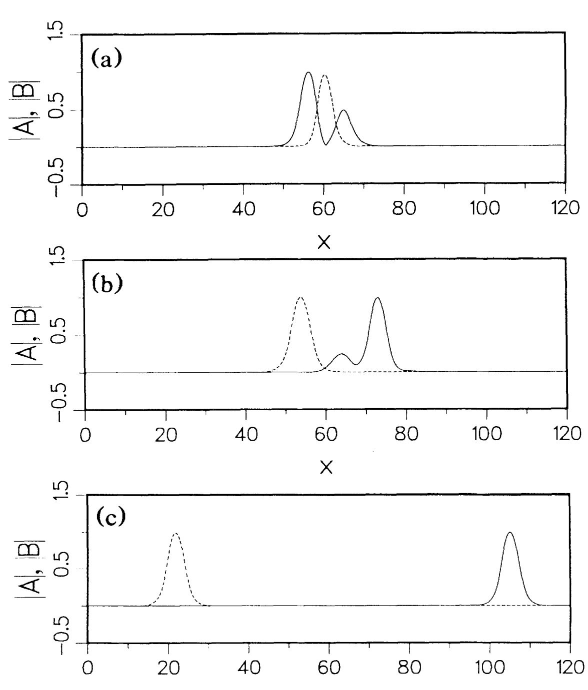

FIG. 5. Partial annihilation of a two-particle state by a one-particle state; identical initial condition as in Fig. 3 ($\xi_r = 1.5$): (a) initial interaction, (b) interaction nearly finished, and (c) final state.

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[16]In all the plots we have chosen the parameter values used in Ref. 13: $\chi = -0.1$, $\beta_r = -3$, $\beta_i = -1$, $\gamma_r = 1$, $\gamma_i = 0$, $\delta_r = 2.75$, $\delta_i = -1$, and we have set all the coefficients of the nonlinear gradient terms to zero. We have also chosen $v = 1$ and $\xi_i = 1$. ξ_r is the only parameter varying from one plot to the next.