Quantization of Self-Dual Field Revisited

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A self-dual field is described by the Lagrangian for an ordinary scalar field with a term added to it to take care of the self-duality constraint. A self-consistent Hamiltonian formulation is obtained using Dirac's method. The constraints are second class, the auxiliary field drops out of the Hamiltonian, and the quantized theory does not show any violation of causality.

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Self-dual fields in two dimensions, sometimes called chiral bosons, are basic ingredients in the formulation of the heterotic string.[1] The quantization of a scalar self-dual field has been greatly discussed recently.[2,3] Siegel's theory[4] with a local symmetry seems to be equivalent[5] to the dimension-zero-field formulation of Floreanini and Jackiw.[2] However, the Euler-Lagrange equations for this field lead to the result that it is the space derivative of the field which satisfies the self-duality condition and not the field itself. The Hamilton equations, on the other hand, being linear, do result in a self-duality condition for the field. Moreover, this field violates the microcausality postulate.[6] No completely satisfactory quantized theory of a self-dual field seems available.[2,3,6]

We propose here to study the quantization, by Dirac's method,[7] of the self-dual field described by the Lagrangian for an ordinary scalar field with a term added to it to take care of the self-duality constraint. The motivation for such a study derives from an analogous situation in Yang-Mills theory:[8] The time component of the vector potential A^0 appears here as an auxiliary (Lagrange multiplier) field. If we decide to choose the gauge $A^0=0$ before varying the action, we miss the first-class Gauss's-law constraint. The Lagrange equations of motion do, however, lead to the vanishing of the time derivative of this constraint and we are required to impose an appropriate boundary condition to work in the right sector. Keeping the A^0 term allows us to derive the Gauss's-law constraint from the Lagrangian and a self-consistent Hamiltonian formulation is obtained by following Dirac's procedure where, if we wish, we may eliminate[8] A^0 by a choice of gauge. For the action of a scalar self-dual field proposed here a similar situation is obtained except that the constraints are second class and the auxiliary field is removed from the reduced Hamiltonian via the Dirac bracket. We show that a self-consistent Hamiltonian formulation can be developed and no violation of causality in the quantized theory occurs.

The Lagrangian for a scalar self-dual field will be taken to be

$$\mathcal{L}=\tfrac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi)+\lambda_\mu(\epsilon^{\mu\nu}+\eta^{\mu\nu})\partial_\nu\phi\,, \tag{1}$$

which is Lorentz invariant and contains a bilinear term in the field ϕ which is dynamical contrary to the auxiliary vector field which appears only linearly. The resulting equations of motion are

$$\partial^\mu\partial_\mu\phi+(\epsilon^{\mu\nu}+\eta^{\mu\nu})\partial_\nu\lambda_\mu=0\,, \tag{2}$$

$$(\epsilon^{\mu\nu}+\eta^{\mu\nu})\partial_\nu\phi=0\,. \tag{3}$$

From the self-duality equation (3) for dynamical field ϕ we derive $\partial^\mu\partial_\mu\phi=0$ and, consequently, from (2) we obtain $(\epsilon^{\mu\nu}+\eta^{\mu\nu})\partial_\nu\lambda_\mu=0$, but not the Klein-Gordon equation $\partial^\mu\partial_\mu\lambda_\mu=0$ for all the components of λ_μ. This field will be seen to be removed from the reduced Hamiltonian obtained by following Dirac's procedure for constrained dynamical systems,[7] leaving behind only the physical dynamical field. It is convenient but not necessary to rewrite (1) in a simpler form as

$$\mathcal{L}=\tfrac{1}{2}[(\partial_0\phi)^2-(\partial_1\phi)^2]+\lambda(\partial_0-\partial_1)\phi\,, \tag{4}$$

where $\eta_{\mu\nu}=\text{diag}(1,-1)$, $\epsilon_{01}=1$, $\lambda=\lambda_0+\lambda_1$. The equations of motion then read as $(\partial_0-\partial_1)\phi=0$, $(\partial_0-\partial_1)\lambda=0$, etc. Denoting by p_λ and $\Pi\equiv\Pi^0=\partial_0\phi+\lambda$ [where $\Pi^\nu=\partial\mathcal{L}/\partial(\partial_\nu\phi)$] the canonical momenta corresponding to λ and ϕ, respectively, the primary weak constraint is $p_\lambda\approx 0$. The canonical Hamiltonian is obtained to be

$$\mathcal{H}_c=\tfrac{1}{2}(\Pi-\lambda)^2+\tfrac{1}{2}(\partial_1\phi)^2+\lambda\,\partial_1\phi\,. \tag{5}$$

By requiring persistence in time of the primary constraint, the secondary constraint follows as $\Phi\equiv\Pi-\partial_1\phi-\lambda\approx 0$, and the extended Hamiltonian $\mathcal{H}'=\mathcal{H}_c+up_\lambda+v\Phi$, where u and v are arbitrary functionals, gives rise to $d\Phi/dt\equiv\{\Phi,H'\}=\partial_1\lambda-(u-2\partial_1 v)$ which allows us to assure through an appropriate choice of u and v that the constraint Φ is preserved in time and no additional constraints arise in the theory. The two constraints are second class as is evident from their Poisson brackets,

$$\begin{aligned}&\{\Phi(x,t),\Phi(y,t)\}=-2\,\partial_x\delta(x-y)\,,\\&\{p_\lambda(x,t),p_\lambda(y,t)\}=0\,,\\&\{\Phi(x,t),p_\lambda(y,t)\}=-\delta(x-y)\,.\end{aligned} \tag{6}$$

The Dirac bracket with respect to these constraints is easily found to be

$$\{f(x,t),g(y,t)\}^* = \{f,g\} + 2\int\int dz\, dz'\, \partial_z \delta(z-z')\{f,p_\lambda(z,t)\}\{p_\lambda(z',t),g\} + \int dz\,[\{f,p_\lambda(z,t)\}\{\phi(z,t),g\} - (\Phi \leftrightarrow p_\lambda)]\,, \quad (7)$$

and we can implement the weak (second-class) constraints now as strong relations, e.g., $p_\lambda = 0$ and $\Pi = \partial_1\phi + \lambda$, e.g., $\Pi_- \equiv \Pi_0 - \Pi_1 = 0$.

The Dirac brackets for the self-dual field are found to coincide with the standard Poisson brackets,

$$\{\phi(x,t),\Pi(y,t)\}^* = \delta(x-y)\,,$$
$$\{\phi(x,t),\phi(y,t)\}^* = \{\Pi(x,t),\Pi(y,t)\}^* = 0\,. \quad (8)$$

The reduced Hamiltonian is then found to be

$$\mathcal{H} = \Pi\,\partial_1\phi\,, \quad (9)$$

which leads to the equations of motion $\partial_0\phi = \partial_1\phi$, $\partial_0\Pi = \partial_1\Pi$ using (8). These are consistent with the Lagrangian formulation and no problem with the causality arises on performing the canonical quantization, $\{f,g\}^* \to -i[f_{op},g_{op}]$. The Lagrangian in the first-order formulation may then be written as ($\Pi_- = 0$)

$$\mathcal{L} = \tfrac{1}{2}\Pi_+\partial_-\phi = \tfrac{1}{2}\Pi_\mu(\eta^{\mu\nu} + \epsilon^{\mu\nu})\partial_\nu\phi\,, \quad (10)$$

which is also conformal invariant. For the anti-self-dual field satisfying $\partial_0\phi = -\partial_1\phi$ we find $\mathcal{L} = \frac{1}{2}\Pi_-\partial_+\phi$. The action for the ordinary field may not, in general, be written as the sum of the actions of these two self-dual fields. In the prescription of canonical quantization[7] operator ordering and hermiticity of quantized operators must be taken care of. We may alternatively use the path-integral formalism due to Batalin, Fradkin, and Vilkowski for a theory with second-class constraints.[9] It is also possible to develop a gauge theory of chiral bosons by adding a Wess-Zumino field in the theory, and the quantization along the lines of Ref. 9 again leads to the action (10).

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