Magnetic Random-Walk Representation for Scalar QED and the Triviality Problem

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A random-walk representation for continuum scalar quantum electrodynamics in the Feynman gauge is derived. The triviality problem of scalar QED is formulated in terms of the triviality of magnetic random-walk interactions. The average partition function z of a pair of magnetic random walks is shown to be equal to 1 for $D \geq 4$.

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(1) *Introduction.*— There is an interesting idea in Euclidean quantum field theory (QFT), which dates back to a famous Symanzik work in 1969,[1] to reformulate QFT models in terms of random walks or "polymer chains." This formulation is important from the point of view of the analysis of triviality phenomena in QFT in the various dimensions D. [A quantum field theory is said to be trivial if its noncutoff limit is a (generalized) free-field theory. The criterion of the triviality is usually given in terms of cumulants (Ursell functions), or more often (weaker criterion) in terms of integrated cumulants ("renormalized coupling constants").] Actually, the idea has been implemented in only a few ($\lambda\phi^4$-like) cases: $\lambda\phi^4$ theory itself, its simple generalizations [e.g., O(N) $\lambda\phi^4$], and some $\lambda\phi^4$-borrowed models (e.g., the nonlinear σ model). The rigorous lattice approach was mainly used;[2] although a more heuristic continuum one is also notable.[3] Probabilistic arguments were given to support the conjectured triviality of $\lambda\phi^4$-like theories for $D>4$ ($D \geq 4$). Triviality in all these models can be heuristically understood through the intersection properties of a pair of Brownian paths.[4]

The aim of this Letter is to extend Symanzik's idea to another QFT model. We would like to propose the simplest and physically interesting model which is by no means related to the $\lambda\phi^4$ one, namely, scalar quantum electrodynamics (QED).[5] As we are going to work in the Euclidean formalism, we will impose the most convenient covariant Feynman gauge. Accordingly, we have a potentially nontrivial theory, which is free of Grassmann objects (fermions and ghosts), and of indices (Abelian gauge group), which might cause troubles in probabilistic interpretation. The possible lack of Symanzik-Nelson positivity could also prevent any probabilistic interpretation of the theory, but it appears that owing to the diamagnetic inequality[6] for regularized scalar fields, which follows from the Kato inequality or from the Itô integral representation, positivity can be asserted. It should be stressed that our approach is not rigorous, and the analysis will be performed in the continuum.

In Sec. 2, we will derive a random-walk (RW) representation for continuum scalar QED in the Feynman gauge for any D.[7] Since the RW's now conduct (fictitious) electric currents, according to the classical Maxwell theory of electromagnetism, one should expect magnetic fields, and possibly magnetic interactions among them as well as with an external current density. So, unlike the $\lambda\phi^4$ case, the RW's, in general, interact via long-range magnetic forces. In order to find the energy of the system, one has to calculate (speaking in terms of classical magnetostatics) the "inductances" for a system of Brownian electric paths. The magnetic energy E for a system of N contours is given by the well-known formula[8]

$$E=\frac{1}{2}\sum_{k,l=1}^{N} L_{kl} I_k I_l , \quad (1)$$

where I_k are the currents, and N is the number of the contours (in our formulation, electric charge plays the role of electric current). Thus, the vanishing of the mutual inductances of the system of RW's is related to the triviality of scalar QED. Since we consider QED in D-dimensional space, the polymer conductors are put also in D-dimensional space, and therefore our magnetostatics is coming from classical electrodynamics in $D+1$ dimensions. As a byproduct of our analysis, we obtain a path-integral ("first-quantized") formulation of scalar QED (strictly speaking, a Euclidean version of it).

In Sec. 3, we will show that the average partition function z for magnetic RW's is equal to 1 for $D \geq 4$, and on this base, we will argue for the vanishing of the mutual inductances. To support our conjecture, we will use the scaling property of Brownian paths, and some simple renormalization-group ideas.

Section 4 concludes our analysis with a discussion devoted to the meaning of the average partition function z, and to the renormalization issue.

(2) *Magnetic random-walk representation.*— According to the plan we have just sketched, first we will derive the magnetic RW representation for the generating functional of continuum scalar QED in the Feynman gauge. The Euclidean action for scalar QED is given by

$$S=\int dx\,(\tfrac{1}{4}F_{\mu\nu}^2+|\nabla_\mu\phi|^2+m^2|\phi|^2) , \quad (2)$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$, and $\nabla_\mu=\partial_\mu-ieA_\mu$ (the $\lambda\phi^4$ self-interaction, usually included for renormalization purposes, is omitted here because it only gives rise to the ordinary δ-like repulsive potential, which demands nonintersecting magnetic RW's). The generating functional has the form

$$Z[J,j]=\int DA\,D\phi^*\,D\phi\exp[-S+(J,A)+(j,\phi)]\,, \qquad (3)$$

where

$$(J,A):=\int dx\,J_\mu(x)A_\mu(x)\,,$$

$$(j,\phi):=\int dx[j^*(x)\phi(x)+j(x)\phi^*(x)]\,.$$

Having imposed the Feynman-gauge condition, we get

$$S=\int dx[-\tfrac{1}{2}A_\mu\Delta A_\mu-\phi^*(\partial_\mu-ieA_\mu)^2\phi+m^2\phi^*\phi]\,. \qquad (2a)$$

Integrating out the fields ϕ^* and ϕ, we get

$$Z[J,j]=\int DA\times\exp\Big[-S_0-\mathrm{Tr}\ln D(A_\mu)+\int dx\,dy\,j(x)D^{-1}(A_\mu)(x,y)j^*(y)+(J,A)\Big]\,, \qquad (4)$$

where

$$S_0:=-\tfrac{1}{2}\int dx\,A_\mu\Delta A_\mu,\quad D(A_\mu):=-(\partial_\mu-ieA_\mu)^2+m^2\,.$$

To express (4) in probabilistic terms, we will introduce the proper-time representation

$$\ln D=\mathrm{const}\times\Big(-\int_0^\infty ds\,s^{-1}\exp(-sD)\Big)\,,\qquad D^{-1}=\int_0^\infty ds\exp(-sD)\,, \qquad (5)$$

and the Feynman-Kac-Itô formula[9]

$$\exp(-\tfrac{1}{2}sD)(x,y)=\exp(-\tfrac{1}{2}m^2s)\int dW(x,y;s)(b)\exp[-V(b)]\,, \qquad (6)$$

where $dW(x,y;s)(b)$ is a conditional Wiener measure for paths $b(t):[0,s)\rightarrow R^D$ which start at $x\in R^D$ and end at $y\in R^D$. In our case, $V(b)$ is given by Itô's integral

$$V(b)=ie\int_0^s A_\mu(b(t))db_\mu(t)+\tfrac{1}{2}ie\int_0^s\partial_\mu A_\mu(b(t))dt\,.$$

Since we drop the constant factor in (5), to normalize $Z[J,j]$ one should replace $Z[J,j]$ by $Z^{-1}[0,0]Z[J,j]$, which introduces some additional $Z_{m,0}$ into (8) below. Using (5) and (6), we will expand (4) into the Taylor series

$$Z[J,j]=\int DA\exp[-S_0+(J,A)]\Bigg[1+\sum_{N=1}^\infty(N!)^{-1}\Big(\int dz\int_0^\infty ds\,s^{-1}\exp(-m^2s/2)\int dW(z,z;s)(b)\exp[-V(b)]+\tfrac{1}{2}\int dx\,dy\,j(x)j^*(y)\int_0^\infty ds\exp(-m^2s/2)\times\int dW(x,y;s)(b)\exp[-V(b)]\Big)\Bigg]\,. \qquad (7)$$

Explicitly, (7) is given by the following expansion:

$$Z[J,j]=Z_0[J]\left[1+\sum_{N=1}^\infty\sum_{m=0}^N(N!)^{-1}\binom{N}{m}Z_{m,N-m}[J,j]\right]. \qquad (8)$$

Here, we have introduced the following notation:

$$Z_{m,n}[J,j]=\int 2^{-n}\left(\prod_{k=1}^m\prod_{l=1}^n dz_k\,dx_l\,dy_l\,j(x_l)j^*(y_l)\right)G_{m,n}(z_1,\ldots,z_m,x_1,y_1,\ldots,x_n,y_n;J)\,, \qquad (9)$$

$$G_{m,n}(z_1,\ldots,z_m,x_1,y_1,\ldots,x_n,y_n;J)=\int_0^\infty\left(\prod_{k=1}^{m+n}\prod_{l=1}^m ds_k\exp(-m^2s_k/2)s_l^{-1}\right)\times\Gamma_{m,n}(z_1,\ldots,z_m,x_1,y_1,\ldots,x_n,y_n,s_1,\ldots,s_{m+n};J)\,, \qquad (10)$$

$$\Gamma_{m,n}(z_1,\ldots,z_m,x_1,y_1,\ldots,x_n,y_n,s_1,\ldots,s_{m+n};J)=\int\left[\prod_{k=1}^m\prod_{l=1}^n dW(z_k,z_k;s_k)(b_k)dW(x_l,y_l;s_{m+l})(b_{m+l})\right]\times\exp[-H_{m+n}(b_1,\ldots,b_{m+n};J)]\,, \qquad (11)$$

$$\exp[-H_N(b_1,\ldots,b_N;J)]=Z_0^{-1}[J]\int DA\exp(-S_0)\exp\left(-\sum_{k=1}^N V(b_k)+(J,A)\right), \qquad (12)$$

where $Z_0[J]$ is the generating functional for pure (free) QED. To perform the final functional integration, we will express the second exponent in (12) in the following manner:

$$-\sum_{k=1}^N V(b_k)+(J,A)=\int dx\,A_\mu(x)\left(J_\mu(x)-ie\sum_{k=1}^N\int_0^{s_k}\delta(x-b_k)db_{k\mu}+\tfrac{1}{2}ie\sum_{k=1}^N\int_0^{s_k}\partial_\mu\delta(x-b_k)dt\right). \qquad (13)$$

Now, we can integrate out the field A_μ, obtaining

$$H_N(b_1,\ldots,b_N;J):=H_N(b,J)=H_N^E(b,J)+H_N^M(b)+H_N^P(b). \quad (14)$$

Here, $H_N^E(b,J)$ denotes the energy of N Brownian electric conductors interacting with the external current density $J_\mu(x)$,

$$H_N^E(b,J)=ie\sum_{k=1}^N\int dx\,J_\mu(x)A_{k\mu}(x)\,, \quad (15)$$

where the electromagnetic potential $A_{k\mu}(x)$ of the kth Brownian conductor is

$$A_{k\mu}(x)=-\int\Delta^{-1}(x,b_k)db_{k\mu}\,.$$

$H_N^M(b)$ is the magnetic energy of a system of N Brownian conductors [compare with (1)],

$$H_N^M(b)=\tfrac{1}{2}e^2\sum_{k,l=1}^N L_{kl}\,, \quad (16)$$

where the inductances L_{kl} are given by the expression

$$L_{kl}=-\int_0^{s_k}\int_0^{s_l}\Delta^{-1}(b_k,b_l)db_{k\mu}db_{l\mu}\,. \quad (17)$$

And finally, $H_N^P(b)$ is the QED polymer energy,

$$H_N^P(b)=\tfrac{1}{8}e^2\sum_{k,l=1}^N\int_0^{s_k}\int_0^{s_l}\delta(b_k-b_l)dt\,dt'\,.$$

For the purely scalar Schwinger functions, the external current density $J_\mu(x)=0$, and hence the energy $H_N^E(b,J)$ vanishes. The polymer energy $H_N^P(b)$ has been investigated in the context of the $\lambda\phi^4$ interactions, and was argued to be trivial for $D\geq 4$. Thus, the possibility of interactions in the purely scalar sector depends only on the inductances L_{kl}.

(3) *Magnetic random-walk interactions.*— Let us now define, for magnetic RW's, the average partition function z ($D\neq 2$):

$$z(s_1,s_2):=\lim_{V\to\infty}V^{-1}\int_V dx\,dW(0;s_1)(b_1)dW(x;s_2)(b_2)\times\exp\left[g\int_0^{s_1}\int_0^{s_2}|b_1-b_2|^{2-D}db_{1\mu}db_{2\mu}\right], \quad (18)$$

where $dW(x;s)(b)$ is a Wiener measure for paths $b(t):[0,s)\to R^D$, which start at $x\in R^D$. The partition function $z(s_1,s_2)$ should satisfy the boundary condition

$$\lim_{s_1,s_2\to 0} z(s_1,s_2)=1\,, \quad (19)$$

and by virtue of the Jensen inequality

$$z(s_1,s_2)\geq 1\,. \quad (20)$$

We can also define the following β function:[4]

$$\beta:=\lim_{\epsilon\to 0}\epsilon^{-1}[z(\lambda s_1,\lambda s_2)-z(s_1,s_2)]=s_1\partial_{s_1}z\,, \quad (21)$$

where $\lambda=1+\epsilon$. According to (18)

$$z(\lambda s_1,\lambda s_2):=\lim_{V\to\infty}V^{-1}\int_V dx\,dW(0;\lambda s_1)(b_1)dW(x;\lambda s_2)(b_2)\exp\left[g\int_0^{\lambda s_1}\int_0^{\lambda s_2}|b_1-b_2|^{2-D}db_{1\mu}db_{2\mu}\right]. \quad (22)$$

Using the Brownian-motion self-similarity, one can rescale the proper times s_1,s_2, and express (22) in the following manner:

$$z(\lambda s_1,\lambda s_2):=\lim_{V\to\infty}V^{-1}\int_V dx\,dW(0;s_1)(b_1)dW(x;s_2)(b_2)\exp\left[g\lambda^{2-D/2}\int_0^{s_1}\int_0^{s_2}|b_1-b_2|^{2-D}db_{1\mu}db_{2\mu}\right]. \quad (23)$$

Applying the Jensen inequality to (23) (for $D\geq 4$), we can remove the power $\chi=\lambda^{2-D/2}$ outside the integral, obtaining the estimation

$$z(\lambda s_1,\lambda s_2)\leq\left[\lim_{V\to\infty}V^{-1}\int_V dx\,dW(0;s_1)(b_1)dW(x;s_2)(b_2)\exp\left[g\int_0^{s_1}\int_0^{s_2}|b_1-b_2|^{2-D}db_{1\mu}db_{2\mu}\right]\right]^\chi=z^\chi(s_1,s_2)\,. \quad (24)$$

Thus, we finally obtain

$$z(\lambda s_1,\lambda s_2)\leq z+\epsilon(2-D/2)z\ln z+O(\epsilon^2)\,; \quad (25)$$

and our β function is estimated from above by β_0,

$$\beta(z)\leq\beta_0(z)\,, \quad (26)$$

where

$$\beta_0(z)=(2-D/2)z\ln z\,. \quad (27)$$

It is evident that $\beta_0(z)$ [and obviously $\beta(z)$] is nonpositive for $D\geq 4$. The general solution of the renormalization-group equation (27) is

$$z=\exp[s_1^{2-D/2}\psi(s_1/s_2)]\,.$$

Taking into account the boundary condition (19), we obtian $\psi=0$, and hence $z=1$. Then, as it follows from (26), $z\leq 1$ for the exact β function, and by virtue of (20) $z=1$.

(4) *Concluding remarks.*— Thus, the partition function z of a pair of magnetic RW's is equal to 1, which strongly suggests the triviality of scalar QED for $D\geq 4$ (at least in the scalar sector). To give more convincing

arguments, one should note that integrated (purely scalar) correlation functions (weaker criterion) consist of z's which are equal to 1; accordingly, the renormalized coupling constants, as they are differences of correlation functions, should be equal to 0. Formally, one can use the Hölder inequalities to isolate from (11) the integrands of particular z's. [The key observation is the following: No power can change the shape of the integrand (it can only change g), and no power can change z (1 to any power is equal to 1).]

One can also give some heuristic explanation to this triviality phenomenon. Namely, Brownian trajectories are very chaotic, and therefore contributions to the magnetic energy coming from neighboring points cancel each other. But for D less than 4, it need not be true because the trajectories are no longer nonintersecting, and hence there may be singular contributions due to the denominator in (23).

For the time being, the analysis is very qualitative as the possible "infinite renormalization" has not yet been taken into account. To begin with, one has to regularize the integrals, and to replace "classical" quantities with cutoff-dependent bare ones. In our nonlattice approach, the regularization can be most easily achieved in the following way. In the case of a scalar field, one has to bound the lower domain of integration with respect to s (the so-called proper-time regularization) in (5), whereas in the case of an electromagnetic field one should somehow regularize the Laplacian in (17). One can observe that the whole analysis does not depend on the integration with respect to s. Thus, the regularization, and consequently the renormalization, in the scalar sector cannot change the conclusion. On the other hand, the regularization in the electromagnetic sector can influence the shape of the interactions of the RW's in (17). Accordingly, there is a logical possibility (though not very likely) that removing both the scalar and electromagnetic cutoffs will restore interactions.

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