Probability Distribution of a Stochastically Advected Scalar Field

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A systematic method is outlined for constructing workable approximations to the joint probability distribution $\mathcal{P}(\psi,\xi)$ of the amplitude ψ and spatial gradient ξ of an active or passive scalar field that is advected by a prescribed isotropically distributed stochastic velocity field and subjected to molecular diffusion. $\mathcal{P}(\psi,\xi)$ is sampled along fluid-element paths, and closure is obtained by taking a multivariate-Gaussian reference field $\psi_0(\mathbf{x})$ and distorting it locally in $\mathbf{x}$ space so that it exhibits the current $\mathcal{P}(\psi,\xi)$.

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The higher statistics associated with turbulence have resisted deduction from the equations of motion. In particular, there is no systematic derivation of the intermittency of small scales that is a striking feature of large-Reynolds-number turbulence or advection of a passive scalar at large Péclet number.

We outline here a closure method for the evolution of $\mathcal{P}(\psi,\xi,t)$, the isotropic joint probability distribution (PD) of scalar-field amplitude ψ and magnitude ξ of its gradient $\boldsymbol{\xi}$ at a point. The analysis encompasses active as well as passive scalar fields. Pope[1] has given a comprehensive review of past work on scalar PD's. Eswaran and Pope[2] present detailed simulation results for passive-scalar statistics in isotropic turbulence. A recent paper by Sinai and Yakhot[3] helped to motivate our work and point to a fruitful line of attack.

The closures are based on a technique for realizing a scalar field $\psi(\mathbf{x},t)$ whose multivariate distribution is accessible and imbeds a given $\mathcal{P}(\psi,\xi,t)$. The evolution of $\mathcal{P}(\psi,\xi,t)$ can then be followed along fluid-element trajectories. We also use extensions of methods that have been successful in forming approximations at the moment level: least-squares realization of stochastic processes under statistical constraints[4,5] and the direct-interaction approximation (DIA) and its relatives.[4-8] In the limit of strain that varies very rapidly in time, our results reproduce some asymptotic behavior obtained by other methods.[9] The methods are adaptable to Navier-Stokes dynamics and to multispecies chemical reactions.

Let a zero-mean scalar field be homogeneously and isotropically distributed in D dimensions and obey

$$\mathcal{D}\psi(\mathbf{x},t)/\mathcal{D}t = Q(\psi(\mathbf{x},t)) + \kappa\nabla^2\psi(\mathbf{x},t) \equiv v(\mathbf{x},t)\,, \quad (1)$$

where $\mathcal{D}/\mathcal{D}t \equiv \partial/\partial t + \mathbf{u}(\mathbf{x},t)\cdot\nabla$ is the substantial derivative, $Q(\psi) = -Q(-\psi)$ describes a one-species chemical reaction, κ is the molecular diffusivity, and $\mathbf{u}(\mathbf{x},t)$ is a stochastic velocity field, independent of $\psi(\mathbf{x},0)$, that obeys $\nabla\cdot\mathbf{u}(\mathbf{x},t) = 0$ ($\mathbf{u} = 0$ if $D = 1$) and has prescribed homogeneous and isotropic statistics. Differentiation of (1) yields

$$\frac{\mathcal{D}\xi_i(\mathbf{x},t)}{\mathcal{D}t} = -u_{j,i}(\mathbf{x},t)\xi_j(\mathbf{x},t) + \frac{\partial Q}{\partial\psi}\xi_i(\mathbf{x},t) + \kappa\nabla^2\xi_i(\mathbf{x},t) \equiv w_i(\mathbf{x},t)\,, \quad (2)$$

where $\boldsymbol{\xi} = \nabla\psi$ and $u_{j,i} = \partial u_j/\partial x_i$. Similar equations can be written for higher spatial derivatives.

Equations (1) and (2) present two kinds of closure problem. First, there is the nonlinearity expressed by the terms in Q and in $\mathbf{u}$. Second, the linear terms in κ couple ψ and $\boldsymbol{\xi}$ to higher-order space derivatives. The Q terms are a challenge for moment closures but they make no difficulty in the present PD closures, whatever the form of Q. The κ terms offer the essential difficulty in constructing PD closures.[1] We address this problem here by constructing $\psi(\mathbf{x},t)$ as a time-dependent nonlinear mapping of a multivariate-Gaussian reference field $\psi_0(\mathbf{x})$, in such a way as to reproduce the current $\mathcal{P}(\psi,\xi,t)$ at each t. The mapping method is nonperturbative and can represent wildly non-Gaussian $\psi(\mathbf{x},t)$. For this reason it may be of general interest in problems where expansions about Gaussian statistics lead to grief. The key feature of the mapping that makes closure possible is that the nth-order space derivatives of ψ are expressed in terms of nth- and lower-order space derivatives of ψ_0. The statistics of the ψ_0 derivatives are known because ψ_0 is Gaussian. This leads to explicit expressions for $\nabla^2\psi$ and $\nabla^2\xi$.

The quantities $v(\mathbf{x},t)$ and $\mathbf{w}(\mathbf{x},t)$ may be written (cf. Ref. 4) as

$$\begin{aligned} v(\mathbf{x},t) &= [v(\mathbf{x},t)]_{C:\psi\xi} + [v(\mathbf{x},t)]_{U:\psi\xi}\,, \\ \mathbf{w}(\mathbf{x},t) &= [\mathbf{w}(\mathbf{x},t)]_{C:\psi\xi} + [\mathbf{w}(\mathbf{x},t)]_{U:\psi\xi}\,, \end{aligned} \quad (3)$$

where $[\cdot]_{C:\psi\xi}$ denotes ensemble mean conditional on given ψ and ξ values (an ordinary nonrandom function

of ψ and ξ) and the remainder $[\cdot]_{U:\psi\xi}$ is uncorrelated with all functions of ψ and ξ.

If $g(\psi,\xi)$ is any function of ψ and ξ, then

$$\int g(\psi,\xi)\mathcal{P}(\psi,\xi)d\psi\, d\xi = \langle g(\psi,\xi)\rangle ,$$

where $\langle\cdot\rangle$ denotes average over the ensemble of realizations of $\psi(\cdot,t)$ and $\mathbf{u}(\cdot,t)$ with ψ and ξ both measured at the same point $\mathbf{x}$. The change in $\langle g(\psi,\xi)\rangle$ during dt induced under (1) and (2) is

$$dt[\langle v(\mathbf{x},t)\partial g/\partial\psi\rangle + \langle w_i(\mathbf{x},t)\partial g/\partial\xi_i\rangle] .$$

The contributions of the $[\cdot]_{U:\psi\xi}$ parts to this expression vanish under the averaging. It follows that the effective flow in (ψ,ξ) space that evolves $\mathcal{P}(\psi,\xi)$ depends solely on the $[\cdot]_{C:\psi\xi}$ parts: The continuity equation for $\mathcal{P}(\psi,\xi)$ is

$$\frac{\partial\mathcal{P}}{\partial t} + \frac{\partial}{\partial\psi}[\mathcal{P}V(\psi,\xi,t)] + \frac{\partial}{\partial\xi}\cdot[\mathcal{P}\mathbf{W}(\psi,\xi,t)] = 0 , \tag{4}$$

where

$$\begin{aligned} V(\psi,\xi) &= Q(\psi) + \kappa[\nabla^2\psi]_{C:\psi\xi} , \\ W_i(\psi,\xi) &= \frac{\partial Q}{\partial\psi}\xi_i + \kappa[\nabla^2\xi_i]_{C:\psi\xi} + Z_i , \end{aligned} \tag{5}$$

and $Z_i = -[u_{j,i}\xi_j]_{C:\psi\xi}$. The $[\cdot]_{U:\psi\xi}$ parts do contribute indirectly by affecting the evolution of the $[\cdot]_{C:\psi\xi}$ parts.

Integration of (4) over ξ gives

$$\frac{\partial\mathcal{P}(\psi)}{\partial t} + \frac{\partial}{\partial\psi}[\mathcal{P}(\psi)\bar{V}(\psi,t)] = 0 , \tag{6}$$

where $\mathcal{P}(\psi)\equiv\int\mathcal{P}(\psi,\xi)d\xi$ and

$$\begin{aligned} \bar{V}(\psi) &= \frac{1}{\mathcal{P}(\psi)}\int\mathcal{P}(\psi,s)V(\psi,s)d\mathbf{s} \\ &= Q(\psi) + \kappa[\nabla^2\psi]_{C:\psi} . \end{aligned} \tag{7}$$

Here $[\cdot]_{C:\psi}$ denotes ensemble mean conditional only on ψ. By use of statistical homogeneity, Eq. (6) can be recast in the form[3,10]

$$\frac{\partial\mathcal{P}(\psi)}{\partial t} + \frac{\partial}{\partial\psi}[Q(\psi)\mathcal{P}(\psi)] = -\frac{\partial^2}{\partial\psi^2}[\chi(\psi)\mathcal{P}(\psi)] , \tag{8}$$

where $\chi(\psi) = \kappa[\xi^2]_{C:\psi}$ is the mean scalar dissipation rate, conditional on given ψ. If $\psi(\mathbf{x},t)$ is multivariate Gaussian at some t, the joint PD of ψ, ξ, $\nabla^2\psi$, and $\nabla^2\xi$ at a point $\mathbf{x}$ is fully determined by the covariances of these quantities. It follows from homogeneity that $\langle\psi\xi\rangle = \langle\xi\nabla^2\psi\rangle = \langle\psi\nabla^2\xi\rangle = 0$ and that

$$\begin{aligned} [\nabla^2\psi]_{C:\psi\xi} &= -\psi\langle\xi^2\rangle/\langle\psi^2\rangle , \\ [\nabla^2\xi]_{C:\psi\xi} &= -\xi\langle(\nabla^2\psi)^2\rangle/\langle\xi^2\rangle . \end{aligned} \tag{9}$$

The variables ψ and $\nabla^2\psi$ are statistically independent of ξ and $\nabla^2\xi$; $[\nabla^2\psi]_{U:\psi\xi}$ and $[\nabla^2\xi]_{U:\psi\xi}$ are Gaussian variables independent of each other and of all the other variables named. Thus $\chi(\psi)$ is independent of ψ.[3,10]

Now let $\psi_0(\mathbf{x})$ be a time-independent, multivariate-Gaussian reference field whose one-point PD is denoted by $\mathcal{P}_0(\psi_0)$. The mapping

$$\psi = X(\psi_0,t) \quad [\partial X/\partial\psi_0 > 0 , \quad X(-\psi_0,t) = -X(\psi_0,t)] \tag{10}$$

will produce a $\mathcal{P}(\psi,t)$ that obeys (6) if

$$\partial X/\partial t = \bar{V}(\psi) = Q(\psi) + \kappa[\nabla^2\psi]_{C:\psi} , \tag{11}$$

so that $\partial X/\partial t$ and $\bar{V}$ describe the same trajectory in ψ space. The initial condition for (6) then is set by $X(\psi_0,0)$. Chain differentiation of (10) and use of the Gaussian relations (9) for the reference-field distribution yield $\xi = \xi_0\partial X/\partial\psi_0$ and

$$\langle\xi^2\rangle = \langle\xi_0^2\rangle\langle(\partial X/\partial\psi_0)^2\rangle , \tag{12}$$

$$[\nabla^2\psi]_{C:\psi} = \langle\xi_0^2\rangle\left(-\frac{\psi_0}{\langle\psi_0^2\rangle}\frac{\partial X}{\partial\psi_0} + \frac{\partial^2 X}{\partial\psi_0^2}\right) , \tag{13}$$

$$[\nabla^2\psi]_{U:\psi} = \frac{\partial X}{\partial\psi_0}[(\nabla^2\psi)_0]_{U:\psi} + \frac{\partial^2 X}{\partial\psi_0^2}(\xi_0^2 - \langle\xi_0^2\rangle) , \tag{14}$$

$$\chi(\psi) = \kappa\langle\xi_0^2\rangle(\partial X/\partial\psi_0)^2 , \tag{15}$$

where the independence of ψ_0 and $\xi_0\equiv\nabla\psi_0$ is used. Equation (14) shows that $[\nabla^2\psi]_{U:\psi}$ is statistically dependent on ψ.

Equations (11)–(13) yield the final closure equation

$$\frac{\partial X}{\partial t} = Q(X) + \kappa\langle\xi_0^2\rangle\left(-\frac{\psi_0}{\langle\psi_0^2\rangle}\frac{\partial X}{\partial\psi_0} + \frac{\partial^2 X}{\partial\psi_0^2}\right) . \tag{16}$$

It is not actually necessary to solve (6) since (10) implies

$$\mathcal{P}(\psi) = \mathcal{P}_0(\psi_0)[\partial X/\partial\psi_0]^{-1} . \tag{17}$$

Equation (10) replaces the true multivariate distribution at each t with a tractable one obtained by local distortion of the field $\psi_0(\mathbf{x})$. In each interval dt, the evolution of this distorted field is treated exactly. Therefore the closure is expected to exhibit the tendency of diffusion to relax non-Gaussian $\mathcal{P}(\psi)$ toward Gaussian form.[1] This physics shows clearly in (16). The two κ terms describe, respectively, an outgoing wave in ψ space and diffusive smoothing. The wave motion expresses the general damping of excitation while the diffusive term gives relaxation toward Gaussian statistics; $\partial^2 X/\partial\psi_0^2$ vanishes if ψ is Gaussian.

The relaxation has previously proved difficult to capture in closures. The underlying reason is suggested by (8), in which the χ term has the form of a *negative* diffusion. The latter expresses the general shrinking of amplitudes induced by κ, but it makes (8) highly unstable in form. Physically acceptable closures must fix $\chi(\psi)$ so as properly to counteract this instability.

Despite its good physics, (16) is a very limited approximation. It takes no account of advective stretching and, more generally, ignores changes in $\langle\xi^2\rangle$ not associated with local rescaling of ψ via (10). External information about second-order moments can be used, if available, to

improve the closure: If $\langle\xi^2\rangle/\langle\psi^2\rangle$, a measure of effective dissipation-scale size, is known, then $\langle\xi_0^2\rangle$ can be reset after each step dt so that (10) and (12) give this $\langle\xi^2\rangle/\langle\psi^2\rangle$. Equation (1) and homogeneity imply $d\langle\psi^2\rangle/dt = 2[\langle\psi Q(\psi)\rangle - \kappa\langle\xi^2\rangle]$. Thus the resetting gives exact evolution of $\langle\psi^2\rangle$ if $Q=0$, in which case it simply rescales t in (16).

Figure 1 compares the evolution of $\mathcal{P}(\psi,t)$ obtained by direct computer simulation of (1) with an integration of (16) for $D=1$, $Q(\psi)=-\alpha\psi|\psi|$, $\mathbf{u}=0$, and Gaussian $\mathcal{P}(\psi,t=0)$ [$X(\psi_0,0)=\psi_0$]. The simulations were started from realizations of ψ at 10^5 points unit spaced over a cyclic line segment, with multivariate-Gaussian statistics and a Gaussian-shaped wave-number spectrum. In the integration of (16), $\langle\xi_0^2\rangle$ was reset after each time step so that the closure gave the same value of $\langle\xi^2\rangle/\langle\psi^2\rangle$ as the simulation. Equation (16) would have overestimated the rate of diffusive effects if the resetting had not been performed. Also shown are the exact evolution for $\kappa=0$ and the prediction of the widely used Gaussian closure obtained by substituting (9) into (7). For that closure also, $\langle\xi^2\rangle/\langle\psi^2\rangle$ was given the simulation value after each time step.

The results highlight the crucial role played by the relaxation property of (16). The Gaussian closure for $\mathcal{P}(\psi,t)$ is shape preserving.[1] The difference between its prediction and the curve for $\kappa=0$ is a measure of quenching of the reaction by the diffusive reduction of amplitudes. Clearly this represents only a small part of the total effect of the κ term in (1).

Successively higher closures may be sought by adjoining to (10) a coordinate transformation that effectively distorts the space in which ψ_0 lives and thereby matches successively larger joint PD's: $\mathcal{P}(\psi,\xi)$, $\mathcal{P}(\psi,\xi,\nabla^2\psi)$, etc. Suppose that the PD's are obtained by sampling over a set of trajectories with large, but finite, number density in $\mathbf{x}$ space. In the neighborhood of each trajectory at time t, take the reference Gaussian field as $\psi_0(\mathbf{z})$, where $\mathbf{z}$ denotes a reference coordinate system with origin on the trajectory. There is more than one way to achieve the needed space distortion. We choose here to relate $\mathbf{z}$ to laboratory Cartesian coordinates (with origin on the trajectory) by the locally isotropic transformation

$$\partial|\mathbf{z}|/\partial|\mathbf{x}| = J(\psi_0,\xi_0,t)\,, \qquad \partial^2|\mathbf{z}|/(\partial|\mathbf{x}|)^2 = K(\psi_0,\xi_0,\nabla^2\psi_0,t)\,, \tag{18}$$

etc. These relations are imposed at the origin and it is understood that space derivatives of ψ_0 are with respect to $\mathbf{z}$; e.g., $\xi_0\equiv\partial\psi_0/\partial z$.

Chain differentiation now yields explicit expressions for $\mathbf{x}$-space derivatives of ψ. We give the results only for the simplest case, in which K and higher functions vanish and (18) describes a ψ_0- and ξ_0-dependent isotropic squeezing or stretching that is uniform in the infinitesimal neighborhood of each trajectory. Equations (10) and (18) then yield

$$\xi = \xi_0 J(\psi_0,\xi_0)\partial X/\partial\psi_0 \equiv \mathbf{Y}(\psi_0,\xi_0)\,, \tag{19}$$

$$[\nabla^2\psi]_{C:\psi\xi} = J^2\left(-\psi_0\frac{\langle\xi_0^2\rangle}{\langle\psi_0^2\rangle}\frac{\partial X}{\partial\psi_0}+\xi_0^2\frac{\partial^2 X}{\partial\psi_0^2}\right), \tag{20}$$

$$[\nabla^2\xi]_{C:\psi\xi} = \xi_0 J^3\left(-\frac{\langle(\nabla^2\psi_0)^2\rangle}{\langle\xi_0^2\rangle}\frac{\partial X}{\partial\psi_0} - \psi_0\frac{D+2}{D}\frac{\langle\xi_0^2\rangle}{\langle\psi_0^2\rangle}\frac{\partial^2 X}{\partial\psi_0^2}+\xi_0^2\frac{\partial^3 X}{\partial\psi_0^3}\right). \tag{21}$$

To complete the closure it is necessary to tie the evolution of X and J to that of $\mathcal{P}(\psi,\xi)$ by relations like (11). This raises a problem. The hierarchical structure whereby X, J, K, etc., have successively more arguments is needed to make nth-order space derivatives of ψ depend only on nth- and lower-order derivatives of ψ_0. In turn, this is crucial to obtaining explicit expressions for $[\nabla^2\psi]_{C:\psi\xi}$, etc. However, while X depends only on ψ, V given by (5) and (20) depends on both ψ and ξ. This problem can be resolved by rewriting (4). It can be verified by substitution that $\partial\mathcal{P}(\psi,\xi)/\partial t$ is unchanged if V and $\mathbf{W}$ in (4) are, respectively, replaced by $\bar{V}$, defined in (7), and

$$W_i'(\psi,\xi) = \xi_i\frac{\xi^{-D}}{\mathcal{P}(\psi,\xi)}\int_\xi^\infty\left[\frac{\partial}{\partial\psi}[\mathcal{P}(\psi,s)\bar{V}(\psi)]+\frac{\partial\mathcal{P}(\psi,s)}{\partial t}\right]s^{D-1}\,ds\,, \tag{22}$$

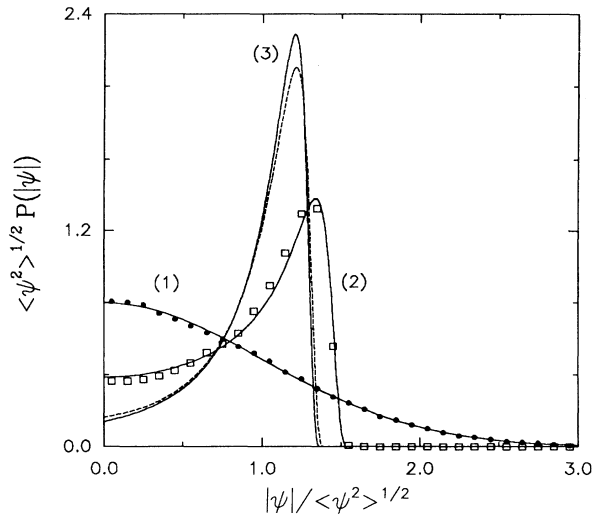

FIG. 1. Diffusive reaction with $\alpha=1$, $\kappa=2.5$, and initial values $\langle\psi^2\rangle=1$, $\langle\xi^2\rangle=0.02$. At $t=4$ (shown), the evolved values were $\langle\xi^2\rangle/\langle\psi^2\rangle=0.01349$, $\langle\psi^2\rangle=0.02329$ (simulation), $\langle\psi^2\rangle=0.02332$ [by (16)], while $\langle\xi_0^2\rangle$ was reset from an initial value of 0.02 to 0.0101. Curve 1, initial Gaussian PD; curve 2, evolved PD according to (16); curve 3, exact evolved PD if $\kappa=0$; dashed curve, evolved PD according to Gaussian closure. The solid (open) data points are the initial (evolved) simulation values.

where $\partial\mathcal{P}(\psi,s)/\partial t$ is given by (4) in its original form.

The mapping defined by (10) and (18) now gives the same trajectory in (ψ,ξ) space that $\bar{V}$ and W' give if

$$\partial X(\psi_0)/\partial t=\bar{V}(\psi)\,,\quad \partial\mathbf{Y}(\psi_0,\xi_0)/\partial t=\mathbf{W}'(\psi,\xi)\,. \tag{23}$$

The initial condition for (4) is then set by $X(\psi_0,t=0)$ and $J(\psi_0,\xi_0,t=0)$. Z_i remains to be evaluated. By (10) and (19),

$$\mathcal{P}(\psi,\xi)=\frac{\mathcal{P}_0(\psi_0,\xi_0)}{(\partial X/\partial\psi_0)(\partial Y/\partial\xi_0)}\,,$$

so that it is unnecessary to integrate (4) explicitly.

The external parameter in the present closure is $\langle(\nabla^2\psi_0)^2\rangle$, instead of $\langle\xi_0^2\rangle$ as in the closure based solely on (10). As before, there are the two possibilities of leaving this parameter fixed throughout the evolution, or using external information to reset it at each t, via the closure equations, so as to give correct evolution of the suitably normalized quantity $\langle(\nabla^2\psi)^2\rangle$. If the sequence of closures proposed in connection with (18) converges usefully, resetting of the external parameter associated with each closure should yield smaller and smaller changes of that parameter with time as closure order increases. The two orders of closure presented above give some support to this hope: If $Q=\mathbf{u}=0$ and $\langle\xi_0^2\rangle$ is kept constant in time, the closure for $\mathcal{P}(\psi)$ based on (10) alone gives the exactly correct decay of $\langle\psi^2\rangle$ and $\langle\xi^2\rangle$ for an initially Gaussian field only if the spectrum support of ψ is confined to a thin shell in k space. But with $\langle(\nabla^2\psi_0)^2\rangle$ kept constant, the closure based on (10) and (18) gives the exactly correct decay of these quantities for a Gaussian field with any self-preserving spectrum shape.

The $\mathbf{u}$ term in (1) can be treated by the DIA or by related approximations to yield moment closures.[5-8] Parallel treatments of the $\mathbf{u}$ term can be constructed for PD's.[11] The DIA result for Z_i in (5) involves $\mathcal{P}(\psi',\xi',t')$, a mean Green's function $G(\psi,\xi,t;\psi',\xi',t')$ for infinitesimal PD perturbations, and the Lagrangian time covariance of $u_{j,i}$. For lack of space, we omit this analysis and give here only the result for a Markovian approximation in which $u_{j,i}$ is made to vary infinitely rapidly in time while keeping its original value for the integral of Lagrangian time covariance over difference time.[12] In this limit, $G(\psi,\xi,t;\psi',\xi',t')$ can be replaced by $\delta(\psi-\psi')\delta(\xi-\xi')$ and Z_i is exactly

$$Z_i=-\theta^L(t)S(t)\xi^2\partial\ln\mathcal{P}(\psi,\xi)/\partial\xi_i\quad(D>1)\,. \tag{24}$$

Here $S(t)=\langle[u_{1,1}(\mathbf{x},t)]^2\rangle$, $\theta^L(t)$ is the original Lagrangian correlation time of $u_{1,1}$, and we use $u_{i,i}=0$. Ordinary second-order perturbation analysis[12,13] also yields (24).

The essential character of Z_i is displayed by taking the limit form (24) and setting $Q=\kappa=0$. Then isotropy relations reduce (4) to the diffusive wave equation

$$\frac{\partial p(\zeta,t)}{\partial t}=\theta^L(t)S(t)\left[\frac{\partial^2 p(\zeta,t)}{\partial\zeta^2}-D\frac{\partial p(\zeta,t)}{\partial\zeta}\right], \tag{25}$$

where $\zeta=\ln(\xi/\xi_{00})$ so that $\xi\partial/\partial\xi=\partial/\partial\zeta$, ξ_{00} is an arbitrary reference value, and $p(\zeta,t)\propto\xi^D\int\mathcal{P}(\psi,\xi,t)d\psi$ is the probability density for ζ.

If the velocity field is statistically stationary, S is independent of time while $\theta^L(t)$ is constant and $O(S^{-1/2})$ for $t-t_0\gg S^{-1/2}$. At such t, (25) yields a normal $p(\zeta,t)$ with mean and variance both $O(S^{1/2}t)$. This corresponds to a highly intermittent (log-normal) distribution for ξ and is consistent with results for rapidly changing shear obtained by other methods.[9] The Eulerian spectrum of passive ψ has been obtained exactly for rapidly varying strain.[12]

The closure based on (10) and (18) can express differential damping effects on ψ due to intermittency of ξ. Thereby it can describe intermittency induced in the ψ distribution as a secondary effect of straining.

Both (10) and (18) satisfy constraints that are exact consequences of homogeneity. They include[14]

$$[\nabla^2\psi]_{C:\psi\xi}=\frac{D}{\mathcal{P}(\psi,\xi)}\int_\xi^\infty\frac{\partial\mathcal{P}(\psi,s)}{\partial\psi}s\,ds\,. \tag{26}$$

If (26) is substituted into (4), integration over ξ and a partial integration with respect to ξ yield (8).

No closure for nonlinear terms is needed when our method is applied to Navier-Stokes dynamics (optionally with advected scalar); all the nonlinear terms are handled like the Q terms in the analysis above. However, the pressure terms must be approximated in terms of single-point statistics by use of the nonlinear mappings and by appeal to constraints based on homogeneity.

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