Dynamical Mass Generation in 3D Four-Fermion Theory

G. W. Semenoff[(1)] and L. C. R. Wijewardhana[(2)]

[(1)]*Department of Physics, University of British Columbia, Vancouver, Canada V6T 2A6*

[(2)]*Department of Physics, University of Cincinnati, Cincinnati, Ohio 45221*

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A mechanism for dynamical generation of Fermion mass in three dimensions is proposed. Spontaneous breaking of parity as well as certain flavor symmetry is discussed.

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Since the seminal work of Nambu and Jona Lasinio,[1] dynamical symmetry breaking has been a cornerstone of modern elementary-particle physics.[2] Chiral-symmetry breaking in QCD is a well-known realization of this phenomenon. It is also proposed as a mechanism for electroweak-symmetry breaking[3] alternative to the conventional approach which uses elementary Higgs fields. It has the appeal of naturalness and requires fewer *ad hoc* assumptions than other approaches.

There are several field-theoretical models in which symmetries are broken by composite order parameters. In this Letter we study a new example of a $D=3$ field theory where nonrenormalizable four-fermion operators drive fermion mass generation for certain critical values of their coupling constants. We shall also use a lattice version of the theory to illustrate the symmetry-breaking mechanism and discuss how our results could be relevant to 2D condensed-matter systems.

In $D=3$, fermion mass can violate parity (P) and time-reversal (T) symmetry: Under P, $(x_0',x_1',x_2')=(x_0,-x_1,x_2)$, a two-component fermion transforms as $\psi(x)\to\gamma_1\psi(x')$ and $\bar\psi(x)\to-\bar\psi(x')\gamma_1$ and the mass operator is odd, $\bar\psi(x)\psi(x)\to-\bar\psi(x')\psi(x')$. Gauge fields also admit local and gauge-invariant but P- and T-violating mass terms, Chern-Simons three-forms.[4,5] Dynamical generation of either P-violating fermion or gauge-field mass could lead to a spontaneous violation of P and T, and generation of mass for one would lead to mass for the other through radiative corrections.[6]

For any pair of fermions it is possible to define a P- and T-invariant mass operator $\bar\psi_1\psi_1-\bar\psi_2\psi_2$ where, as well as the spacetime transformation, P and T interchange ψ_1 and ψ_2. This mass term breaks a Z_2 symmetry which interchanges ψ_1 and ψ_2 and a global flavor symmetry which rotates ψ_1 into ψ_2. Dynamical P-invariant mass generation in 3D QED with large N has been studied using the Dyson-Schwinger gap equation for the fermion mass.[7] It has been found that the flavor-symmetry-breaking pattern $U(2N)\to U(N)\times U(N)$ occurs only when $N<N_c\approx 4$.[8] Preliminary analysis indicates that electromagnetic interactions suppress P-violating mass generation.[9] In the following we reconsider P and flavor-symmetry breaking in large-N QED with additional four-fermion couplings.

This work has direct implications for physical systems where dynamics are confined to a plane, particularly certain 2D condensed-matter systems that can be modeled by relativistic $D=3$ field theory. Examples are the P- and T-symmetric large-N flux phases[10-12] of the Hubbard-Heisenberg model which have a continuum effective action resembling massless spinor electrodynamics. We shall argue that four-fermion couplings arise naturally in those models. An important question arising in their application to high-T_c superconductors is whether there is a further breaking of P and T.[13] There are other condensed-matter systems, such as the 2D graphite model,[14] where the low-energy electron spectrum is relativistic and our analysis would apply.

Consider N four-component fermions in $D=3$ Euclidean space with the action[15]

$$S=\int d^3x\,[A\bar\psi i\gamma_\mu\partial_\mu\psi+iB\phi\bar\psi\psi+iC\chi\bar\psi\tau\psi+(N\lambda B^2/2A^2)\phi^2+(N\kappa C^2/2A^2)\chi^2]\,. \quad (1)$$

[Eliminating ϕ and χ with equations of motion yields four-fermion vertices $(A^2/2\lambda N)(\bar\psi\psi)^2$ and $(A^2/2\kappa N)\times(\bar\psi\tau\psi)^2$.] γ_μ are Hermitian, 4×4, and block diagonal, $\gamma_\mu=\mathrm{diag}(\sigma_\mu,\sigma_\mu)$; σ_μ are Pauli matrices; and $\tau=\mathrm{diag}(1,-1)$. A,B,C are cutoff-dependent constants. This model has $U(N)\times U(N)\times Z_2\times P$ symmetry. The coupling constants λ^{-1} and κ^{-1} have dimension $[\mathrm{mass}]^{-1}$ and the conventional perturbation expansion is nonrenormalizable. It has been conjectured that the $1/N$ expansion is renormalizable.[16] There is a recent rigorous construction where it was shown that with a bare fermion mass the four-fermion operator is relevant at fixed points of the couplings $1/\lambda_c, 1/\kappa_c$.[17] Furthermore, in an interesting series of papers[18] four-fermion theories in $D=3$ were argued to have phases which break Z_2 and were used to study spontaneous breaking of a continuous chiral symmetry. We consider (1) an effective field theory with cutoff Λ and study both P and Z_2 breaking for which the pseudoscalar ϕ and the scalar χ are order parameters. The scalar two-fermion vertices and fermion propagator are finite as $\Lambda\to\infty$ if

$$A=a\left(1-\frac{4}{3\pi^2N}\ln\frac{\Lambda}{\mu}\right)\approx a\left(\frac{\mu}{\Lambda}\right)^{4/3\pi^2N},$$

$$B=C=b\left(1+\frac{4}{\pi^2N}\ln\frac{\Lambda}{\mu}\right)\approx b\left(\frac{\Lambda}{\mu}\right)^{4/\pi^2N},$$

where μ is an arbitrary dimensional parameter. A change of μ can always be compensated by a change in a and b. The effective potential for the classical fields ϕ and χ at leading and next-to-leading order in $1/N$ is

$$\Gamma=\frac{N\lambda B^2}{2A^2}\phi^2+\frac{N\kappa C^2}{2A^2}\chi^2 -N\int\frac{d^3p}{(2\pi)^3}\ln\det\left[\gamma_\mu p_\mu+i\frac{B\phi}{A}+i\tau\frac{C\chi}{A}\right] +\tfrac{1}{2}\int\frac{d^3p}{(2\pi)^3}\ln\det\begin{bmatrix}\lambda-\Delta_{\phi\phi} & -\Delta_{\phi\chi}\\ -\Delta_{\chi\phi} & \kappa-\Delta_{\chi\chi}\end{bmatrix},$$

where the last term is the sum of ring diagrams. With

$$\Delta(p,m)=\frac{\Lambda}{\pi^2}-\frac{|m|}{2\pi}-\frac{p^2+4m^2}{4\pi p}\arctan\left[\frac{p}{2|m|}\right],$$

Δ_{ab} are $\Delta_{\phi\phi}=\Delta_{\chi\chi}=\Delta(p,(B/A)\phi+(C/A)\chi)+\Delta(p,(B/A)\times\phi-(C/A)\chi)$ and $\Delta_{\phi\chi}=\Delta_{\chi\phi}=\Delta(p,(B/A)\phi+(C/A)\chi)-\Delta(p,(B/A)\phi-(C/A)\chi)$ are components of the one-loop scalar self-energy. In leading order in $1/N$

$$\Gamma=N\frac{b^2}{a^2}(\lambda-\lambda_c)\frac{\phi^2}{2}+N\frac{b^2}{a^2}(\kappa-\kappa_c)\frac{\chi^2}{2} +\frac{N}{6\pi}\frac{b^3}{a^3}(|\phi+\chi|^3+|\phi-\chi|^3),$$

where $\lambda_c^0=2\Lambda/\pi^2=\kappa_c^0$. For sufficiently strong coupling $1/\lambda>1/\lambda_c$ or $1/\kappa>1/\kappa_c$ and sufficiently large N, P, or Z_2 is spontaneously broken. In next-to-leading order there are logarithmic corrections. The quadratic terms in ϕ, χ are

$$\Gamma=N\frac{B^2}{A^2}\left[(\lambda-\lambda_c)\left(1-\frac{8}{\pi^2N}\ln\frac{\Lambda}{4(\lambda-\lambda_c)}\right)-(\kappa-\kappa_c)\frac{8}{\pi^2N}\ln\frac{\Lambda}{4(\kappa-\kappa_c)}\right]\frac{\phi^2}{2} +(\lambda,\phi\leftrightarrow\kappa,\chi)+\cdots,$$

where $\lambda_c=\kappa_c=(2\Lambda/\pi^2)(1-2/N)$. The remainder is finite as $\Lambda\to\infty$, of order $|\phi|^3,|\chi|^3$ for large ϕ,χ, and vanishes faster than quadratic order as $\phi,\chi\to 0$. With

$$\lambda-\lambda_c=\frac{A^2}{B^2}\left[\left(1+\frac{8}{\pi^2N}\ln\frac{\Lambda}{\mu}\right)t+\frac{8}{\pi^2N}\tilde{t}\ln\frac{\Lambda}{\mu}\right],$$

$$\kappa-\kappa_c=\frac{A^2}{B^2}\left[\left(1+\frac{8}{\pi^2N}\ln\frac{\Lambda}{\mu}\right)\tilde{t}+\frac{8}{\pi^2N}t\ln\frac{\Lambda}{\mu}\right]$$

(where a change in the parameter μ would be compensated by a change in $t,\tilde{t}$),

$$\Gamma=N\left[t\left(1-\frac{8}{\pi^2N}\ln\frac{\mu}{4t}\right)-\frac{8}{\pi^2N}\tilde{t}\ln\frac{\mu}{4\tilde{t}}\right]\frac{\phi^2}{2} +N\left[\tilde{t}\left(1-\frac{8}{\pi^2N}\ln\frac{\mu}{4\tilde{t}}\right)-\frac{8}{\pi^2N}t\ln\frac{\mu}{4t}\right]\frac{\chi^2}{2}+\cdots$$

is finite as $\Lambda\to\infty$. For fixed $\tilde{t}$, P breaks at $t=(8\tilde{t}/\pi^2N)\ln(\mu/4\tilde{t})$ and the scalar correlation is

$$\approx\text{const}\times|t-(8\tilde{t}/\pi^2N)\ln(\mu/4\tilde{t})|^{1+8/\pi^2N},$$

and the critical exponent is $1+8/\pi^2N$.

To couple electrodynamics to (1) we add

$$S_{\rm em}=\int[A\bar{\psi}\gamma_\mu\mathcal{A}_\mu\psi+(N/2A^2e^2)\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu}].$$

The photon self-energy is $\Pi_{\mu\nu}=(p^2\delta_{\mu\nu}-p_\mu p_\nu)\Pi_e(p)+\epsilon_{\mu\nu\lambda}p_\lambda\Pi_o(p)$. With

$$\frac{\pi_e(p,m)}{N}=-\frac{|m|}{4\pi|p|^2}-\frac{p^2-4|m|^2}{8\pi|p|^3}\arctan\left[\frac{p}{2|m|}\right],$$

$$\frac{\pi_0(p,m)}{N}=\frac{m}{2\pi|p|}\arctan\left[\frac{p}{2|m|}\right],$$

we have

$$A^2\Pi_e=\pi_e\left[p,\frac{B(\phi+\chi)}{A}\right]+\pi_e\left[p,\frac{B(\phi-\chi)}{A}\right],$$

$$A^2\Pi_o=\pi_o\left[p,\frac{B(\phi+\chi)}{A}\right]+\pi_o\left[p,\frac{B(\phi-\chi)}{A}\right].$$

The coefficient of the dynamically generated Chern-Simons term in the effective action is

$$\Pi_o(0)=(N/4\pi A^2)[\mathrm{sgn}(\phi-\chi)+\mathrm{sgn}(\phi+\chi)].$$

It vanishes when $|\chi|>|\phi|$. The electromagnetic corrections to the term quadratic in ϕ and χ in Γ are (in a gauge where electromagnetic fermion wave-function renormalization vanishes)

$$\Gamma_{\rm em}=\tfrac{1}{2}\int^\Lambda\frac{d^3p}{(2\pi)^3}\ln\left\{p^2\left[\frac{1}{e^2}+\Pi_e(p)\right]^2+\Pi_o(p)^2\right\} =\frac{B^2}{A^2}\left[-\frac{e^2}{4\pi^3}\ln\frac{16\Lambda}{e^2}(\phi^2+\chi^2)+\frac{e^2}{4\pi^2}\phi^2\right]+\cdots.$$

Here the first term derives from Π_e and the second from Π_o. The second term resists spontaneous breaking of P. Thus the electromagnetic interactions favor breaking Z_2 instead of P, in accord with Ref. 9. The corrected critical couplings are

$$\lambda_c=\frac{\Lambda}{\pi^2}\left[1-\frac{2}{N}\right]+\frac{e^2}{4\pi^2N}\left[\frac{1}{\pi}\ln\frac{16\Lambda}{e^2}-1\right],$$

$$\kappa_c=\frac{\Lambda}{\pi^2}\left[1-\frac{2}{N}\right]+\frac{e^2}{4\pi^3N}\ln\frac{16\Lambda}{e^2}$$

so that $1/\lambda_c>1/\kappa_c$. This difference is significant if e^2 is of order Λ.

To obtain intuition for how Z_2 and P are broken in the four-fermion theory we consider a Hamiltonian lattice model which produces (1) in the continuum limit. We use N flavors of fermions on a square lattice with spacing s and kinetic energy

$$H_f=\frac{1}{2s}\sum_{\langle x,y\rangle}(\Delta_{xy}^*\psi_x^\dagger\psi_y+\Delta_{xy}\psi_y^\dagger\psi_x), \qquad (2)$$

where $\langle x,y\rangle$ denotes nearest neighbors and the density is N particles per site. Equation (2) describes the effective electron dynamics in the flux phase of the large-N [where spin SU(2) is generalized to SU(N)] mean-field theory of the Hubbard-Heisenberg model.[10,11] In that work $\Delta_{xy}=\langle\psi_x^\dagger\psi_y\rangle$ is the expectation value of the hopping amplitude and is complex. The P- and T-invariant flux phase has Δ with a constant magnitude and a phase that gives magnetic flux π per plaquette. Here we shall also use that background field Δ and see that it yields continuum fermions with a relativistic spectrum. We can later couple gauge fields by making the phase of Δ a dynamical variable and introducing kinetic and potential-energy terms. To go to the continuum limit we divide the lattice, $\mathbf{x}=sn_1\hat{1}+sn_2\hat{2}$ where $\{\hat{1},\hat{2}\}$ are unit basis vectors of the plane, into four sublattices where the integers n_1 and n_2 are separately either even or odd, and label the fermions

$$\psi_1\equiv\psi(\text{even,even}),\quad \psi_2\equiv\psi(\text{even,odd}),$$
$$\psi_3\equiv\psi(\text{odd,even}),\quad \psi_4\equiv\psi(\text{odd,odd}).$$

We choose the background field with $\Delta_{12}=\Delta_{21}=\Delta_{34}=\Delta_{43}=i$, $\Delta_{13}=\Delta_{42}=-1$, $\Delta_{31}=\Delta_{24}=1$. The Hamiltonian is $H_f=(1/s)\int d^2k\,\psi^\dagger(k)M(k)\psi(k)$, where

$$M(k)=\begin{pmatrix}0 & \sin(sk_2) & -i\sin(sk_1) & 0\\ \sin(sk_2) & 0 & 0 & i\sin(sk_1)\\ i\sin(sk_1) & 0 & 0 & \sin(sk_2)\\ 0 & -i\sin(sk_1) & \sin(sk_2) & 0\end{pmatrix}$$

and k is in the Brillouin zone of the even-even sublattice. This Hamiltonian has twofold degenerate eigenvalues $sE_k=\pm[\sin^2(sk_1)+\sin^2(sk_2)]^{1/2}$. The branches intersect at $k=(0,0)$. With a half-filled band the effective Hamiltonian for low-energy electrons is

$$H_f\approx\int d^2k(\Psi_a^\dagger,\Psi_b^\dagger)\begin{pmatrix}\sigma_1k_1+\sigma_2k_2 & 0\\ 0 & \sigma_1k_1+\sigma_2k_2\end{pmatrix}\begin{pmatrix}\Psi_a\\ \Psi_b\end{pmatrix},$$

where $k\in R^2$, σ_i are Pauli matrices, and $\Psi_a=(1/\sqrt{2})\times(\psi_1+\psi_4,\psi_2+\psi_3)$, $\Psi_b=(1/\sqrt{2})(\psi_2-\psi_3,\psi_1-\psi_4)$. This is the $D=3$ Dirac Hamiltonian with N species of four-component fermions. On the lattice there is U(N) flavor symmetry. The lattice doubling,[19] which is responsible for the appearance of four-component fermions, leads to an apparent U($2N$) symmetry. Discrete symmetries are Z_2 (translation by $s\hat{2}$ and $\psi_1\to-\psi_1$, $\psi_2\to-\psi_2$) and P [reflection through the line $\mathbf{x}=(s/2)\hat{1}$ and then the Z_2 transformation]. The P-even, Z_2-odd mass is a local operator on the lattice [$\gamma_0=\text{diag}(\sigma_3,\sigma_3)$, $\tau=\text{diag}(1,-1)$], $\bar{\Psi}\tau\Psi=(\psi_1^\dagger\psi_1+\psi_4^\dagger\psi_4-\psi_2^\dagger\psi_2-\psi_3^\dagger\psi_3)$, and its expectation value measures the asymmetry in charge distribution between the $n_1+n_2=$even and $n_1+n_2=$odd sublattices. Z_2-symmetry breaking is a formation of a site-centered charge-density wave. A lattice operator that produces the Euclidean four-fermion operator $(1/2N\kappa)\times(\bar{\psi}\tau\psi)^2$ is $H_\chi=-(1/2sN\kappa)\sum_{\langle x,y\rangle}(\psi_x^\dagger\psi_x-\psi_y^\dagger\psi_y)^2$ which is attractive and favors clumping of fermions N to a site. The critical value of κ arises when the attraction is sufficiently strong to balance the tendency of the kinetic term to spread charge out. This interaction is contained in the attractive Hubbard interaction for spin-$\frac{1}{2}$ fermions, $H_{\text{Hub}}=-U\sum\psi_\uparrow^\dagger\psi_\downarrow^\dagger\psi_\downarrow\psi_\uparrow$. Appropriately generalized, SU(2)$\to$SU(N),[11]

$$H_{\text{Hub}}=-\frac{U}{N}\sum_x(\psi_x^\dagger\psi_x)^2$$
$$=-\frac{U}{N}\sum_{\langle x,y\rangle}[(\psi_x^\dagger\psi_x-\psi_y^\dagger\psi_y)^2+(\psi_x^\dagger\psi_x+\psi_y^\dagger\psi_y)^2].$$

The first term produces the P-invariant mass in the continuum limit. The second term is a nonrelativistic charge-density–charge-density interaction. In the case relevant to high-T_c superconductors,[11] the Hubbard interaction is repulsive and Z_2 would not be broken.

On the lattice the P-odd mass is $\bar{\Psi}\Psi=\psi_1^\dagger\psi_4+\psi_4^\dagger\psi_1-\psi_2^\dagger\psi_3-\psi_3^\dagger\psi_2$ and its expectation value measures an asymmetry in the amplitude for electron hopping diagonally across plaquettes between next-to-nearest neighbors. The corresponding four-fermion operator can be obtained from the lattice interaction Hamiltonian

$$H_\chi=-\frac{1}{8sN\lambda}\sum_x(\psi_{x+\hat{1}+\hat{2}}^\dagger\psi_x+\psi_x^\dagger\psi_{x+\hat{1}+\hat{2}}-\psi_{x+\hat{2}}^\dagger\psi_{x+\hat{1}}-\psi_{x+\hat{1}}^\dagger\psi_{x+\hat{2}})^2.$$

Though this operator is gauge invariant in the continuum it is not so on the lattice and in a gauge theory it would be necessary to introduce next-nearest-neighbor link operators to define it correctly. It is contained (along with other four-fermion operators) in the next-nearest-neighbor Heisenberg antiferromagnet $H_{\text{AF}}=(J/4s)\times\sum_{\text{NNN}}\psi_x^\dagger\boldsymbol{\sigma}\psi_x\cdot\psi_y^\dagger\boldsymbol{\sigma}\psi_y$.

The tight-binding fermion problem on a honeycomb lattice[14] also has a Lorentz-invariant continuum limit. There, four-fermion interactions similar to H_χ and H_ϕ above would arise in a natural way. In a realistic condensed-matter system there are other four-fermion operators which respect the $\text{U}(N)\times Z_2\times P$ (with $N=2$) symmetry but are not Lorentz invariant. However, as we have seen, four-fermion operators can only be relevant at special critical values of their coupling constants. It could well be that there exists a realistic model where the only relevant operators are those which are Lorentz invariant in the continuum. Also, a lattice regularization of the continuum relativistic field theory could be defined in this way. We have demonstrated that four-fermion operators can break P dynamically. This would lead to an induced Chern-Simons term for the gauge field[6] and the charged particles in this theory would have fractional spin and statistics with statistics parameter $1/2+1/N$.[20] This would be particularly interesting in light of recent speculation that even a perfect gas of fractional-statistics particles has a superconducting ground state.[21] We believe that the P-breaking phase that we find coincides with the chiral spin states that have recently been suggested as candidates for high-T_c superconductor ground states.[12]

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