Steady-State Distribution of Generalized Aggregation System with Injection

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We consider the kinetics of a steady-state aggregation process in which the basic dynamical variable is a "charge" which can assume both positive and negative values. We rigorously show that the charge distribution follows a power law in the steady state which is sustained by random injection. The exponent of the power law depends both on the type of injection and on the spatial dimension.

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An aggregation system typically involves the irreversible joining of diffusing particles whenever they meet. Since such an irreversible process is a typical nonthermal equilibrium phenomenon, we expect that the study of the statistical properties of aggregation is a basic step toward the construction of a statistical physics for systems far from thermal equilibrium. Considerable work has already been performed on the statistical properties of aggregation, and it is acknowledged that a steady state which is quite different from thermal equilibrium can be realized by the continuous injection of small particles.[1] For example, for many aggregating systems in the steady state it is known that the cluster mass distribution follows a power law with infinite variance.[2,3]

In this paper, we generalize steady-state aggregation to include the possibility that the dynamical variable can have both positive and negative values.[4] Intuitively we can interpret each cluster having a positive or negative "charge" which is conserved when two particles collide. We study the steady-state properties of this generalized system for the following two types of injection: (1) random positive and negative injection and (2) random pair creation injection. It is theoretically shown that the distribution of the charge m in the steady state has a power-law tail for any type of injection in one dimension and in the mean-field case. The exponent of the power law depends both on the type of the injection and on the spatial dimension.

First, we describe the model. We consider the aggregation process in a discretized space and time. On every site there is at most one particle. If more than two particles happen to hop onto one site, they immediately coalesce into a single particle with the charge of the product equal to the sum of the charges of the two incident particles. Let $m(j,n)$ be the charge of the particle on site j at the nth time step. The aggregation process can be represented by the following stochastic equation for $m(j,n)$:

$$m(j,n+1)=\sum_k W_{jk}(n)m(k,n)+I(j,n)\,, \tag{1}$$

where $I(j,n)$ denotes the charge injected at the jth site at time n, and $W_{jk}(n)$ is a stochastic variable which is equal to 1 when the particle on the kth site jumps to the jth site and which is equal to 0 otherwise. Since one particle cannot go to two different sites in a single time step, $W_{jk}(n)$ must be normalized as $\sum_j W_{jk}(n)=1$. In the following theoretical analysis we consider the two simple cases: (A) $W_{jk}(n)=1$ with probability $\frac{1}{2}$, $W_{jk-1}(n)=1$ with probability $\frac{1}{2}$, and $W_{jk}(n)=0$ for $k\neq j,j-1$; (B) $W_{jk}(n)=1/N$ with probability $1/N$, where N is the total number of sites, which will be taken to infinity in our theoretical analysis. Case (A) with periodic boundary condition corresponds to the situation of aggregating Brownian particles in one dimension if we observe the system from a coordinate which moves with a constant mean velocity of $\frac{1}{2}$. Case (B) corresponds to the mean-field limit.

The distribution of m in case (A) can be obtained by introducing the r-body characteristic function

$$Z_r(\rho,n)\equiv\left\langle \exp\left[i\rho\sum_{j=1}^{r} m(j,n)\right]\right\rangle, \tag{2}$$

where $\langle\cdots\rangle$ denotes the average over all realization of $\{W_{jk}(n)\}$. Assuming that the distribution of injection is independent and identical, we have the following evolution equation for Z_r from Eq. (1):

$$Z_r(\rho,n+1)=\left\langle \exp\left[i\rho\sum_{j=1}^{r}\sum_k W_{jk}(n)m(k,n)+i\rho\sum_{j=1}^{r} I(j,n)\right]\right\rangle=\Phi(\rho)^r\{Z_{r+1}(\rho,n)+2Z_r(\rho,n)+Z_{r-1}(\rho,n)\}/4\,, \tag{3}$$

where $\Phi(\rho)\equiv\langle\exp[i\rho I(j,n)]\rangle$ is the characteristic function for the injection process which can be expanded as $\Phi(\rho)=1+i\langle I\rangle-\langle I^2\rangle\rho^2/2+\cdots$.

In the steady state we have the following set of linear equations for $Z_r(\rho)$, $r=1,2,3,\ldots$, with the boundary condition $Z_0=1$:

$$Z_{r+1}(\rho)+[2-4\Phi(\rho)^{-r}]Z_r(\rho)+Z_{r-1}(\rho)=0\,. \tag{4}$$

Neglecting higher-order terms of ρ in $\Phi(\rho)$ we can show that Eq. (4) becomes identical to the following Bessel function's recurrence relation:

$$J_{n+1}(x)-2nJ_n(x)/x+J_{n-1}(x)=0\,, \qquad (5)$$

by taking $x=1/[-2i\langle I\rangle\rho-\langle I^2\rangle\rho^2]$ and $n=r-1/x$. Then, the solution of Z_1 is evaluated using the properties of Bessel functions as

$$Z_1(\rho)=\begin{cases}1-c\langle I\rangle^{1/3}|\rho|^{1/3}i^{-1/3}+\cdots, & \text{for } \langle I\rangle\neq 0, \quad (6a)\\ 1-c\langle I^2\rangle^{1/3}|\rho|^{2/3}2^{-1/3}+\cdots, & \text{for } \langle I\rangle=0\,, \quad (6b)\end{cases}$$

where c is a numerical constant.[5] Since the characteristic function is the Fourier transform of the probability density $p(m)$, we obtain the charge distribution by inversion. In the case where $\langle I\rangle>0$ (or $\langle I\rangle<0$) we have

$$p(m)\propto m^{-4/3}, \text{ for } m\gg\langle I\rangle \text{ (or for } m\ll\langle I\rangle)\,, \qquad (7a)$$

while in the case $\langle I\rangle=0$ we have

$$p(m)\propto m^{-5/3}, \text{ for } |m|\gg\langle I^2\rangle^{1/2}. \qquad (7b)$$

Equation (7a) shows a one-sided power law whose exponent agrees with that of the known distributions for positive variables.[3] Equation (7b) gives a symmetric power law which is a new property of the "charge aggregation."

Next we consider pair creation injection where a positive and a negative charge are injected simultaneously but randomly at nearest-neighbor sites. Mathematically the injection is given by $I(j,n)=I$, $I(j+1,n)=-I$. It is easy to show that the many-body characteristic function satisfies the following set of equations instead of Eq. (4) in the steady state:

$$Z_2(\rho)+[2-4\Phi(\rho)^{-1}]Z_1(\rho)+1=0\,, \qquad (8a)$$

and for $r=2,3,4,\ldots,$

$$Z_{r+1}(\rho)+[2-4\Phi(\rho)^{-2}]Z_r(\rho)+Z_{r-1}(\rho)=0\,. \qquad (8b)$$

Equations (8a) and (8b) can be transformed into a quadratic equation for $Z_1(\rho)$, and by solving it we get the following symmetric charge distribution:

$$p(m)\propto|m|^{-2}, \text{ for } |m|\gg\langle I^2\rangle^{1/2}. \qquad (9)$$

Note that the exponent of -2 is the same as that of a Lorentzian.

In the mean-field case (B), we can derive the following equation for the time evolution for both types of injection processes:

$$Z_1(\rho,n+1)=\Phi(\rho)\sum_{r=0}^{N}a_rZ_1(\rho,n)^r\,, \qquad (10)$$

where $a_r\equiv\binom{N}{r}(1/N)^r(1-1/N)^{N-r}$. In the limit $N\to\infty$, the steady-state solution of Eq. (10) is obtained as

$$Z_1(\rho)=\begin{cases}1-\sqrt{2}\langle I\rangle^{1/2}|\rho|^{1/2}i^{-1/2}+\cdots & \text{for } \langle I\rangle\neq 0, \quad (11a)\\ 1-\langle I^2\rangle^{1/2}|\rho|+\cdots & \text{for } \langle I\rangle=0. \quad (11b)\end{cases}$$

Here the pair creation injection is included in Eq. (11b) which gives the same distribution as Eq. (9). Equation (11a) gives the following solution for $m\gg\langle I\rangle>0$ (or for $m\ll\langle I\rangle<0$):

$$p(m)\propto|m|^{-3/2}\,, \qquad (12)$$

which agrees with the known solution[3] in the case where m can take only positive values.

The exponents of the power laws are summarized in Table I. We have Lorentzian distributions both in one dimension and in the mean field for pair creation injection, while for uncorrelated injection the exponents for one dimension and mean field differ. These results suggest the possibility that the critical dimension of the charge aggregation system depends on the types of injection. It can be estimated to be 1 or less for the pair creation injection and to be 2 or larger for the uncorrelated random injection.

As we have seen so far our generalized aggregation model has no control parameter and in the steady state it automatically chooses a power-law distribution which can be considered as a kind of critical behavior. We may regard our model as a new type of the self-organized criticality proposed by Bak, Tang, and Wiesenfeld.[6] Bak's model is composed of threshold elements and it shows critical behaviors in space and time. Our model shows a critical behavior in the distribution of the field variable, m. Both of these two models are very simple and seem to be complementary. We might expect that the notion of self-organized criticality can be deepened by analyzing and combining these two basic models.

It should be noted here that the steady state of our model is a little different from ordinary steady states. We can easily show from Eq. (3) or (10) that the variance, $\langle m^2\rangle-\langle m\rangle^2$, grows always linearly with time. This is a pitfall which might make us doubt the existence of the steady state itself; however, there surely exists the steady state in which the variance is divergent. We cannot only prove the stability of the steady-state distribution obtained precedingly, but also calculate the relaxation to the steady state.[7]

Because of this divergence an intuitive understanding is possible to the question why we always have a power-law distribution in the steady state of our model. In the theory of stable distributions[8] the central-limit theorem

TABLE I. The power exponent of the steady-state distribution for the one-dimensional case (1D) and the mean-field case (mf). The distribution is symmetric when the mean value of the injection, $\langle I\rangle$, is equal to zero including the case of pair creation injection (PC).

	$\langle I\rangle\neq 0$	$\langle I\rangle=0$	PC
1D	$\frac{4}{3}$	$\frac{5}{3}$	2
mf	$\frac{3}{2}$	2	2

is generalized to independent random variables with divergent variance. It is known that the limit distribution of the sum of such variables is, if exists, a non-Gaussian stable distribution which has a long tail of power. In our model the random variable m can be decomposed into a sum of aggregated variables which also have divergent variances. Although we cannot directly apply the generalized central-limit theorem to our model because the aggregated variables may have some mutual dependence, it seems likely that the limit distribution of m belongs to a non-Gaussian stable distribution and thus has a power tail.

Relating to the problem of the dependence, we can show that the mutual dependence of the field variables is very weak even in one dimension. It is already known in the case of constant injection[9] that the r-body characteristic function, $Z_r(\rho)$, is nearly equal to the independent case, $[Z_1(\rho)]^r$, in the vicinity of $\rho=0$. The same relation also holds for the generalized injection cases. An interesting consequence of this result is that the distribution of the sum of r charges, $m_1+m_2+\cdots+m_r$, has the same power tail. As a result the variance of the total charge in an interval of finite length is always divergent. This extraordinarily large spatial fluctuation of charge distribution might be related to the segregation phenomenon which has been found for two species annihilation system with injection.[10]

The potential applicability of our model is expected to be very wide. By regarding $m(i)$ as the height difference in a unit length we can consider a random surface problem in one dimension.[11] The aggregation process makes two dislocations coalesce and the injection creates a new dislocation or a bump (pair creation). The above steady state corresponds to the situation that these smoothing and roughening effects balance.

Another potential field of application may be turbulence. It is well known that one-dimensional turbulence governed by Burgers equation is described by a set of shock waves which behave just like sticky particles in the low-viscosity limit.[12] Relations between three-dimensional turbulence and aggregation phenomenon have not been clarified yet, but a kind of aggregation process might be also relevant in three dimensions. The author and a co-worker have already analyzed a vectorized version of the aggregation process, the vortex-tube aggregation.[13] We have numerically shown that there appears a statistically steady state when we keep injecting vortex rings at random, and found a Lorentzian distribution of circulation in the steady state. Experimentally it is known that the relative velocity distribution of fully developed stationary turbulence also follows a Lorentzian.[14] Since the circulation is expected to be roughly proportional to the relative velocity, this coincidence might suggest that three-dimensional turbulence actually has a deep relation to the steady state of generalized aggregation with injection.

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