Analytical Progress towards the Mass Spectrum and Deconfining Temperature in SU(3) Gauge Theory

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Using Lüscher's small-volume expansion, the mass spectrum and deconfining temperature of SU(3) gauge theory are evaluated. Including nonperturbative features by restoring symmetries which were broken in perturbation theory we obtain results which are valid up to intermediate volumes. The mass spectrum obtained is in good agreement with the perturbative results for small volumes, and with Monte Carlo data in medium-sized volumes. Using asymmetric volumes we are able to estimate the deconfining temperature and find reasonable agreement with Monte Carlo calculations.

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In this Letter I describe an approach which is based on a small-coupling expansion of the continuum theory in a finite box pioneered by Lüscher.[1] The expansion obtained this way is an expansion in the physical volume of the system. Defining the theory in a box will produce a discrete momentum spectrum of the fields and, as long as the spacing is large enough (i.e., the box is small enough), one is allowed to treat higher momentum modes as a perturbation to the lower modes. One can define an effective Hamiltonian in the modes with momentum smaller or equal than some momentum k_0 by integrating out all modes with a momentum higher than k_0 perturbatively, and use standard quantum mechanics to derive the mass spectrum of the effective Hamiltonian.[2,3] The choice of $k_0=0$, i.e., integrating out all nonzero-momentum modes, is the easiest one, but we are still left with a multidimensional effective Hamiltonian [9 dimensions for SU(2) and 24 for SU(3)].

't Hooft[4] defined electric and magnetic flux in the box in a gauge-invariant way and was able to relate the electric flux to a group of symmetry transformations, called central conjugations, of the Hamiltonian which is isomorphic to the center Z_n of SU(n). These transformations divide the Hilbert space into n^3 subspaces and states with nonzero electric flux transform nontrivial under central conjugations. For small coupling g the wave functions are concentrated around the classical vacuum and its central conjugations. For larger g tunneling between these different sectors sets in (i.e., the wave function is nonzero for the gauge configurations lying between the vacuum and its central conjugation) and the energy is no longer degenerate with respect to the electric flux.

The effective Hamiltonian of Lüscher is not symmetric under the central conjugations. A way to restore the symmetry and to define electric flux has been pioneered by Koller and van Baal[5,6] (with an amendment by the present author[7]) for the case of SU(2). This process can be described by using the effective Hamiltonian in each of the toron sectors (i.e., the sectors obtained by expanding about the vacuum or any of its central conjugations) and extending the support of the wave functions to the region which is obtained by patching the sectors together. The wave functions are required to be continuous across the sector boundaries. The effective Hamiltonian is then symmetric under central conjugations and we can construct the different sectors of electric flux by enforcing the transformation laws for different units of electric flux on our wave functions. However, this process can be done only for certain gauge configurations, namely the vacuum valley (see, for example, Refs. 6 and 8). It is reasonable, however, to assume that the transformation laws obtained this way are a good estimate for the transformations on all gauge configurations. Using this approach and extending it to SU(3) we are able to obtain the glueball spectrum in intermediate volumes.

Another point of interest is the deconfining phase transition in SU(3). However, this study is difficult to perform in symmetric volumes, because one would require very large volumes to overcome finite-size effects, which shift the critical temperature. Instead of using a symmetric torus we will exploit an asymmetric geometry[9,10] and use finite-size scaling theory[11-13] to relate "small-volume" results to the infinite-volume limit. In this way we obtain an estimate of the SU(3) deconfining transition.

The details of the calculation will be described in a forthcoming publication.[8]

The results for the mass spectrum are obtained by considering a symmetric box of size l^3. Compared to SU(2) we have an additional quantum number for the glueball spectrum, which is the transformation under charge conjugation C. Our states are described by the irreducible representation of the cubic group, their parity, and the charge parity. The mass estimates and the energy of the electric flux are obtained by subtracting the vacuum energy (i.e., the lowest A_1^{++} from the energy eigenvalues of the effective Hamiltonian). All ener-

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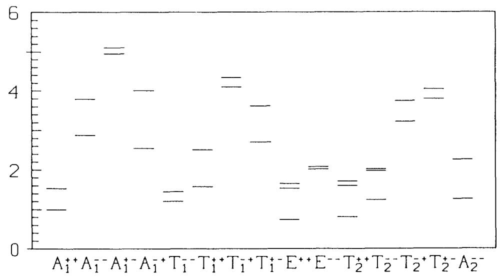

FIG. 1. Mass ratios $m/m(A_1^{++})$ in the perturbative regime at $g=0.6$. The lowest states of the $A_2^{--}, T_1^{--}, T_2^{--}$ representations form a 3^{--} multiplet. The second lowest states of the $A_1^{++}, T_2^{++}, E^{++}$ and the lowest state of the T_1^{++} form a 4^{++} multiplet.

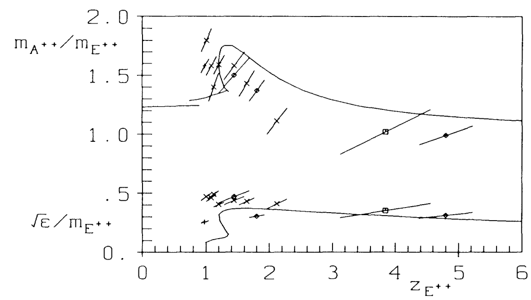

FIG. 2. Mass ratios and Monte Carlo results. The values for the mass ratio up to $z=1$ are the perturbative results of Ref. 3 and the Monte Carlo data are from Ref. 14.

gies given below were calculated using the Raleigh-Ritz variational technique with a basis of 1500 wave functions, which were selected for each coupling g out of a set of approximately 30000000 wave functions. The SU(3) irreducible representations (0,0) through (4,4) were used; however, some wave functions involving (4,4) were not taken into account. For $PC=-1$ the wave functions vanish on the vacuum valley and we were not able to derive the boundary conditions for this case and carried over the results from the $PC=+1$ sectors. Therefore, the interpretation of the results for the $PC=-1$ sectors is doubtful beyond the tunneling transition. We estimate the errors of our masses due to the finite basis to be of the order of (2–5)% for the $PC=+1$ states. For the sectors with $PC=-1$ the SU(3) representations that contribute to the wave states are reduced to (1,1), (2,2), and (3,3) and we expect the error to be of the order of (10–20)%.

In order to compare our results to the perturbative calculation found in Ref. 3 we plot the mass ratios $m/m(A_1^{++})$ for various representations of the cubic group at $g=0.6$. This value for g is large enough to avoid the failure of our basis for small g, but still small enough to avoid effects of tunneling, which are not included in Ref. 3. The ratios are given in Fig. 1. We find good agreement with the results of Ref. 3. To compare our results with Monte Carlo results[14,15] we use the Fisher scaling variable z, which expresses the physical size of the system and is defined by $z=ml$, where m is a typical mass of the system and l is the size of the system. In Fig. 2 we plot the mass ratio $m(A_1^{++})/m(E^{++})$ and the ratio $\sqrt{\varepsilon}/m(E^{++})$ of the square root of the "string tension" and the E^{++} mass versus $z_{E^{++}}=m(E^{++})l$. In contra- distinction to SU(2) we do not have a smooth behavior of the A_1^{++}, E^{++} mass ratios during the onset of tunneling. Although the absolute value of the masses and the onset of tunneling are slightly off, the mass ratios themselves agree well with Monte Carlo results. The sharp crossover of the $m(A)/m(E)$ mass ratio explains the seeming contradiction between the perturbative results[3] and the Monte Carlo data, which were only taken in the symmetric phase.

Some other mass ratios are given in Fig. 3. Analogous to the case of SU(2) the T_2^{++} is degenerate with the E^{++} for small g and becomes twice as heavy beyond the tunneling transition. The E^{++} and A_1^{++} are then the lowest masses in the system. If we were able to make our box larger and larger, we would encounter at some point the restoration of the O(3) rotational symmetry. At this point the E^{++} and the T_2^{++} representations should combine to $J^{PC}=2^{++}$ multiplets. It is reasonable to assume that the lowest E and T_2 states will combine to the lowest 2^{++} multiplet.

Monte Carlo results[15] give as an estimate for the T_2 before the symmetry restoration ($z_{A_1^{++}}\approx 7$)

$$\frac{m(T_2^{++})}{m(A_1^{++})}=1.7\,, \qquad (1)$$

which is identical to our result at $z_{A_1^{++}}=7$.

Further, Monte Carlo calculations predict as an estimate for large volumes

$$\frac{m(2^{++})}{m(0^{++})}=1.5\,, \qquad (2)$$

which lies between the E^{++} and T_2^{++} masses in the intermediate-volume case.

The T_2^{--}, the lower T_1^{--}, and an A_2^{--} (not shown in Fig. 3) form in the perturbative regime a 3^{--} multi-

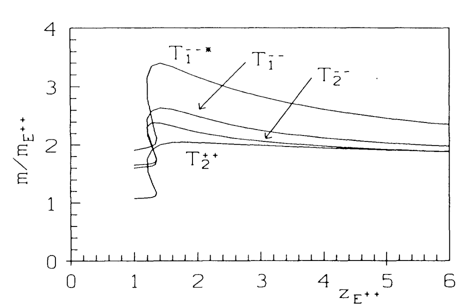

FIG. 3. Mass ratios for some other low-lying states.

plet. It is not clear in which way the states will recombine after the second crossover to J^{PC} multiplets. In one scenario a 3^{--} is formed from the same states as in the perturbative region, and the next higher T_1^{--} is a 1^{--} state. On the other hand, the lowest T_1^{--} can become a 1^{--} in the large-volume limit and the next higher T_1^{--} may combine with the T_2^{--} and the A_2^{--} to a 3^{--} multiplet.

To study the phase transition we follow the approach of Ref. 10 and consider the theory in a box of size $l_t l^2 \infty$, $l = z_t l_t$, with l_t being the (Euclidean) time direction and $z_t \geq 1$. The phase transition will give a signal in the spectrum at some specific g_c, which we can convert with the help of the Callan-Symanzik equation to the physical temperature T_c in units of the scaling parameter Λ.

Because our system is of finite size it will not have a real phase transition, but a crossover from one phase to another. In general, the crossover will not occur at the critical temperature of the infinite system, but one can try to relate the signal of a system of size z to the infinite-volume limit by means of the finite-size scaling theory of phase transitions.

For $g \neq g_c$ the large-l behavior of the energy of the 't Hooft electric flux ϵ is

$$\epsilon(l,g) = \epsilon(\infty,g)[1 + O(e^{-c_1(g)l^2})] \text{ for } g > g_c \tag{3}$$

and

$$\epsilon(l,g) = O(e^{-c_2(g)l^2}) \text{ for } g < g_c , \tag{4}$$

where c_1, c_2 are positive and monotonically increasing in $|g - g_c|$ for g close to g_c. Consider two volumes, $l_t l^2 \infty$ and $l_t l'^2 \infty$, with $l' > l$. The time direction is l_t and the infinite direction is the third spatial direction. From Eqs. (3) and (4) we see that it is possible to find a coupling $g_0(l,l')$ with

$$l'\epsilon(l',g) > l\epsilon(l,g) \text{ for } g > g_0(l,l') \tag{5}$$

and

$$l'\epsilon(l',g) < l\epsilon(l,g) \text{ for } g < g_0(l,l') . \tag{6}$$

This implies

$$l'\epsilon(l',g_0) = l\epsilon(l,g_0) \tag{7}$$

and from Eqs. (3) and (4) it follows further that[16]

$$\lim_{l \to \infty} g_0(l,l') = g_c . \tag{8}$$

In Fig. 4 we plot $z_t \epsilon l_t$ for $z_t = 1.2, 1.4, \ldots, 2.0$.

For the three largest z_t values we obtain a fixed point at $g = g_c = 2.0$, which gives for the critical temperature $T_c \approx 3.5\Lambda_{MS}$ (MS denotes the minimal-subtraction scheme). Using the relation[17,18] between Λ_{MS} and $\Lambda_{lattice}$ this is converted to

$$T_c \approx 38\Lambda_{lattice} . \tag{9}$$

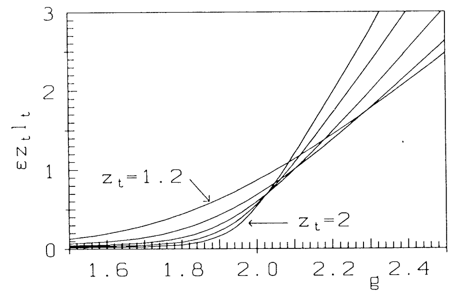

FIG. 4. Energy of the electric flux times z_t for asymmetric volumes.

Recent Monte Carlo results in the scaling region yield[19]

$$T_c \approx 47\Lambda_{lattice} . \tag{10}$$

Thus we find a reasonable agreement between the two approaches.

At this point let me remark two things. The interpretation of the shortest direction as the time direction and the infinite direction as a spatial direction will give another meaning to the electric flux. From 't Hooft's duality relation[4] it follows that a unit of "electric flux" defined in the time direction corresponds to a mixture of systems which all have one unit of electric flux, but in addition also have magnetic flux. Another point[20] is that the approximation we are doing may introduce an error which is more significant for the deconfinement transition than for the mass spectrum.

In summary, we find reasonable agreement between the analytical calculation and the Monte Carlo results in intermediate volumes. The fact that the mass of the 2^{++} lies between the E^{++} and T_2^{++} allows us to speculate that the main ingrediant to reach large volumes is indeed the spatial O(3) symmetry restoration, whereas other effects play a minor role. This view is further substantiated by the agreement of the two methods with respect to the deconfining temperature: The only "ingredient" in the analytical calculation is the tunneling transition. This is already enough to obtain the correct order of magnitude for T_c.

We conclude that the perturbative approach is suited to describe the intermediate-volume range of gauge theories. Unfortunately, for the mass spectrum it can only make very vague predictions for large volumes, because it is not (yet) capable of restoring the O(3) rotational symmetry and we have to view our particles as being still squeezed in too small a box. An advantage of the analytical approach is certainly that we are able to understand the dynamics better than in a numerical approach.

In order to improve the analytical calculation and extend it to larger volumes, we would have to tackle the following problems. The use of a one-loop calculation for g up to ≈ 2.5 is somewhat questionable and we should include higher-order corrections. The more pressing problem, however, is the nonzero-momentum modes. For masses of the order of $ml = 10$, they start to give relevant contributions to the eigenstates and perturbative treatment is no longer justified. An exact treatment of the modes would achieve rotational-symmetry restoration and allow a correct treatment of the boundary conditions. Whereas the former could give rise to the large-volume mass spectrum, the latter is of importance for the deconfining transition, because tunneling is enhanced in two directions compared to a symmetric volume. But already the exact treatment of the first nonzero-momentum mode would give rise to a $3\times 24 = 72$-dimensional Hamiltonian in SU(3), which makes this problem nearly impossible. It should, however, be feasible to treat the first nonzero-momentum modes exactly for the case of SU(2). Finally, one can include fermions[21,22] in the approach, which could give, even for small volumes, a testing ground for lattice calculations.

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