## Exact Fractal Dimension of 2D Ising Clusters

In a recent Letter,[1] Stella and Vanderzande argued that in two dimensions the fractal dimension of Ising clusters at the critical point should be $D_F = \frac{187}{96}$. Ising clusters, which are connected spins of the same sign, are distinct from Ising droplets in models of correlated site-bond percolation.[2] The above authors used the Potts lattice-gas version of these models[3,4] to describe its expected phase diagram,[3,4] and conjecture that the Ising critical point is a tricritical point in this wider space of the $Q=1$ Potts model. Hence, from the known exact value of the Potts tricritical magnetic exponent[4,5] $x_H = \frac{5}{96}$ they get $D_F \equiv 2 - x_H$. In this Comment, we derive the above Ising-cluster fractal dimension by a Coulomb-gas technique applied *directly* to the $O(n=1)$ model at its critical point. Second, we stress again[6,7] that the *geometrical* equivalence between the $O(n)$ loop model and the $Q$-state Potts model applies, not only for $Q=1$ and $n=1$, but for any $n=\sqrt{Q}$, $Q \in [0,4]$, $n \in [0,2]$, between the *low-temperature* (dense) phase of the $O(n)$ model and the $Q=n^2$ *critical* Potts model, or between the *critical* $O(n)$ model and the *dilute tricritical* Potts system.

We start from the Ising model on the triangular lattice $\mathcal{T}$ (Fig. 1). In its *low*-temperature expansion, the clusters of up or down spins have natural boundaries which are nonintersecting loops on the dual hexagonal lattice $\mathcal{H}$. These loops are also graphs of the *high*-temperature expansion of the $O(n)$ loop model[8] defined on $\mathcal{H}$ by

$$Z_{O(n)} = \sum_{\mathcal{G}} W_{O(n)}(\mathcal{G}) = \sum_{\mathcal{G}} K^B n^L , \qquad (1)$$

where the sum runs over graphs $\mathcal{G}$ on $\mathcal{H}$ made of nonintersecting loops, with total numbers of bonds $B$ and of loops $L$. Equation (1) for $n=1$ thus reconstructs the low-$T$ expansion of the triangular Ising model with coupling constant $J$ if $K \equiv e^{-2J/T}$; the critical point occurs at[8] $K_c = 1/\sqrt{3}$. We define for $X,Y \in \mathcal{T}$ the correlator[5]

$$G_{O(n)}(X-Y) = \sum_{\mathcal{G}} W_{O(n)}(\mathcal{G}) \mathbf{1}(X,Y) , \qquad (2)$$

where $\mathbf{1}(X,Y)$ is the characteristic function of a cluster, i.e., $\mathbf{1}=1$ if $X$ and $Y$ are in the same connected component, and $\mathbf{1}=0$ if they are separated by *at least* one loop of the $O(n=1)$ model. To calculate (2), the model (1) is transformed into a solid-on-solid (SOS) model.[7,8] Height variables $\varphi$ are defined on the centers of the hexagons, such that two adjacent heights are equal or differ by $\pm\pi$. The loops of the $O(n)$ model, once *arbitrarily oriented,* become SOS domain walls with a step of $+\pi$ on the left of any oriented line. The weight $W_{\rm SOS}$ is the product along the walls of local factors $Ke^{iu}$ ($Ke^{-iu}$) at each left- (right-) turning vertex. An oriented loop on $\mathcal{H}$ has a total of left minus right local turns always equal to $\pm 6$, hence a phase factor $e^{\pm 6iu}$. Summing over orientations yields $Z_{\rm SOS} \equiv Z_{O(n)}$ for $n = 2\cos(6u)$.[8] We define then the SOS correlator[8,9]

$$G_{\rm SOS}(X-Y, e_1, e_2) \equiv \sum_{\mathcal{G}} W_{\rm SOS}(\mathcal{G}) e^{ie_1\varphi(X) + ie_2\varphi(Y)} . \qquad (3)$$

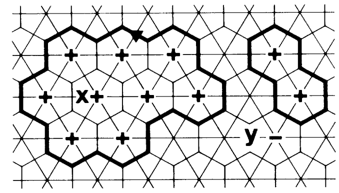

FIG. 1. Clusters of $\pm$ spins surrounded by loops on the dual lattice.

The $O(n)$ weights for loops are reconstructed[9] if the "electric charges" $e_1, e_2$ satisfy $e_1 + e_2 = 0, 2e_0 (\mathrm{mod}\,2)$, where[8] $e_0 \equiv -6u/\pi$. Furthermore, the characteristic function $\mathbf{1}$ in (2) is obtained if we set in (3) $e_1 - e_0 = \pm\frac{1}{2} (\mathrm{mod}\,2)$ and $e_2 - e_0 = \pm\frac{1}{2} (\mathrm{mod}\,2)$. The solution is $e_1 = e_0 + \frac{1}{2} (\mathrm{mod}\,2)$, $e_2 = e_0 - \frac{1}{2} (\mathrm{mod}\,2)$ such that (2) is identical to (3). At $K = K_c$ (or $K > K_c$) $G_{\rm SOS}$ (3) appears as the correlation function of two electric charges, $G_{\rm SOS}(X-Y, e_1, e_2) = |X-Y|^{e_1e_2/g}$, where[7,8] $g$ is the Coulomb-gas coupling constant parametrizing $n = -2 \times \cos(\pi g)$, $e_0 = \pm(1-g)$, with $g \in [1,2]$ at $K_c(n)$ and $g \in [0,1]$ in the dense phase $K > K_c$. Hence the leading cluster exponent is $x = -e_1e_2/2g = 1/8g - (1-g)^2/2g$. The *fractal dimension of the interior of the loops* in the $O(n)$ model is therefore $D_F = 2 - x = 1 + g/2 + 3/8g$. For the critical Ising model $n=1$, $g = \frac{4}{3}$ and we get $D_F = \frac{187}{96}$. Notice that for percolation, i.e., the dense $n=1$ model[6] ($g = \frac{2}{3}$), we recover $D_F = \frac{91}{48}$. Note that $D_F \le 2$ requires $n \in [0,2]$ such that $\frac{1}{2} \le g \le \frac{3}{2}$ with the upper limit $D_F = 2$ for self-avoiding walks ($n=0$), dilute ($g = \frac{3}{2}$) or dense ($g = \frac{1}{2}$). Lastly, in the Potts model, the magnetic exponent reads[5,8] $x_{H,\rm Potts} = 1/2g' - (4 - g')^2/8g'$, where $\sqrt{Q} = -2\cos(\pi g'/4)$, with $g' \in [2,4]$ or $g' \in [4,6]$ for the critical or tricritical Potts point, respectively. We check the forementioned $x_{O(n)} \equiv x_{H,\rm Potts}$ for $g \equiv g'/4$, i.e., $n = \sqrt{Q}$.

B. Duplantier and H. Saleur
Service de Physique Théorique de Saclay
91191 Gif-sur-Yvette CEDEX, France

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