Unambiguous Fermionic-String Amplitudes

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We show how to treat boundary divergences in heterotic string theory covariantly and unambiguously. The method applies even to theories with nonvanishing tadpoles; in this case the Fischler-Susskind mechanism suffices to ensure well defined answers. Also n-point functions are well defined with no special string-tension renormalizations. As an example we find the loop corrections to the linearized background equations of motion for the $O(16)\times O(16)$ string needed to give unambiguous, finite scattering amplitudes. No splitting or projection of supermoduli space is needed.

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One way to quantize a field theory is to quantize small fluctuations about a classical ground-state configuration. If we inadvertently choose a starting configuration which is not an extremum of the classical potential, various pathologies arise in the resulting perturbation theory. In particular, we find that some "particle" states can disappear into the vacuum. If moreover some of these states are massless, then physical amplitudes will diverge due to processes in which a state propagates for a very long time before disappearing. Even if we begin at the classical minimum, loop effects can destabilize the vacuum, giving rise to the same artificial infrared divergences.

In a theory with nonvanishing tadpoles it thus becomes necessary to cut off diagrams which in real space have long on-shell lines, and to introduce compensating shifts in the background in a way which also depends on the cutoff. If all goes well the total, modified theory will have a good limit as the cutoff is removed.

In string theory many of the same considerations apply. Things are not as simple as in the field theory case, however, since it is not so clear how to implement the necessary cutoff. In a multiloop diagram what exactly is the "length" of a given internal tube? There is in fact no coordinate-invariant answer to this question. We must instead introduce into a given diagram some extraneous, coordinate-noninvariant information and hope that the final answer will somehow be independent of this choice.

Fischler and Susskind carried out this program in the bosonic string[1] (see also Ref. 2). By introducing a world-sheet metric one can cut off the one-loop diagram; the resulting metric dependence in the theory then cancels against another dependence, which arises when the background fields are deformed from a conformally invariant false ground state (flat spacetime) to a conformally *noninvariant* true ground state (de Sitter space). Many new results and extensions followed. In particular, Callan, Lovelace, Nappi, and Yost and Polchinski and Cai found that a similar mechanism worked for the type-I superstring to one loop.[3] We will argue below, however, that the full subtlety of the problem is not apparent from this case.

The case of closed fermionic strings has remained unclear. In the picture-changing formalism[4] an apparent ambiguity was found, associated with the boundary of supermoduli space.[5-7] In particular, the problem seemed to be an inherent pathology associated to superspace integration.[6,8] It was found to be harmless in the case of certain theories having no tadpoles, though even in such cases it appeared necessary to renormalize the string tension by hand for the various n-point loop amplitudes.[9] Since ultimately we are interested in theories which break their supersymmetry, it is desirable to extend this result; while presumably a successful string theory will break supersymmetry nonperturbatively, still it is important to understand the perturbative situation first. Similarly one cannot expect to get the corresponding issue in string field theory straight without understanding the first-quantized situation.

In this Letter we will argue that tachyon-free heterotic string theories satisfying a certain condition do not suffer from any ambiguity in perturbation theory. The same mechanism used by Fischler and Susskind, as interpreted by Polchinski,[2] suffices to get well defined amplitudes from such theories.[10] Our argument does not require any facts about the global structure of supermoduli space nor indeed any other new mathematical results. As in Ref. 2, we will see how the operator formalism for string theory greatly clarifies the issues; some of the necessary ingredients for the fermionic case appear in Ref. 12, while a general reference is 13.

As an example we will discuss the $O(16)\times O(16)$ heterotic string,[14] obtaining the linearized corrections to its background equations of motion. Many more details will appear in Ref. 15. Bowing to tradition we will leave a careful treatment of multiple divergences to future work. We expect that similar considerations apply to type-II superstrings as well. Our approach was suggested in the conclusion of Ref. 9 and in Ref. 16. Other authors have proposed approaches to the ambiguity problem which seem very different.[17]

To put the issues in focus, suppose we are given a volume form μ on the complex plane of the form $\mu = |q|^{-2} dq \wedge d\bar{q} + (\text{regular})$, where q is the standard complex coordinate. The integral of μ over the unit disk diverges. We can try defining $I_\epsilon = \int_{|q| > \epsilon} \mu$, but we must keep in mind that this depends not only on ϵ and the form μ, but also on the choice of coordinate q used to exclude an ϵ ball. If $q' = aq + bq^2 + \cdots$ is another coordinate centered at zero then we can define I'_ϵ by excluding $\{|q'| < \epsilon\}$; then $I'(\epsilon) = I(\epsilon) - 4\pi i \ln|a| + O(\epsilon)$. Thus if we propose to find a "counterterm" to add to $I(\epsilon)$ so as to get a good limit as $\epsilon \to 0$, the former must also depend on the same choice of q in a way which cancels the dependence in $I(\epsilon)$. Note that as the cutoff is removed it is only the magnitude of a, the first Taylor coefficient, which matters; were μ more divergent we would have needed more coefficients.

Now suppose there are several more variables, $\vec{m}$, and $\mu = |q|^{-2} dq \wedge d\bar{q} \wedge \Omega + (\text{regular})$, where Ω is some form in $\vec{m}$. Now $I(\epsilon)$ is the integral excluding a cylinder $\{|q| < \epsilon\}$. Once again we can expand $q' = a(\vec{m})q + O(q^2)$, and once again all that matters as $\epsilon \to 0$ is the function $|a(\vec{m})|$. In particular, if we can exclude a cylinder about the locus $\Delta \equiv \{q = 0\}$ in a way which is canonical, or standard, up to first derivatives then we have a canonical way to cut off the integral of μ. If, however, Δ is some complicated surface then in general there will be no natural cutoff prescription and we have to make choices.

In a string theory without tachyons the string measure behaves like μ above. The dangerous locus $\{q \sim 0\}$ consists of surfaces with very long tubes, or equivalently very narrow necks. If the theory has nonvanishing tadpoles it is these configurations which give rise to the divergence described above. The mechanism proposed in Ref. 1 for canceling the divergence works as follows. One chooses a local pinching coordinate q and cuts off the path integral over surfaces of genus $g_1 + g_2$. Then one perturbs the background fields by an amount $\delta\phi_{\text{BG}}^{(g_2)}$ of order λ^{g_2}, where λ is the string coupling. The amplitude to order $\lambda^{g_1+g_2}$ now consists of the original part plus a part obtained by inserting $\delta\phi_{\text{BG}}^{(g_2)}$ into a surface of genus g_1. There are two other contributions as well, shown in Fig. 1.

At first Fig. 1 seems paradoxical. The counterterms I_1, I_3 know nothing about the right-hand side of the original diagram, while I_2, I_3 know nothing about the left-hand side. How then can $I_1 + I_2 + I_3$ cancel the cutoff dependence of I, which in general depends jointly on the shape of *both* sides? The answer is that in general they cannot.[18] If we choose the pinching coordinate q wisely, however, we can nevertheless obtain a good prescription.

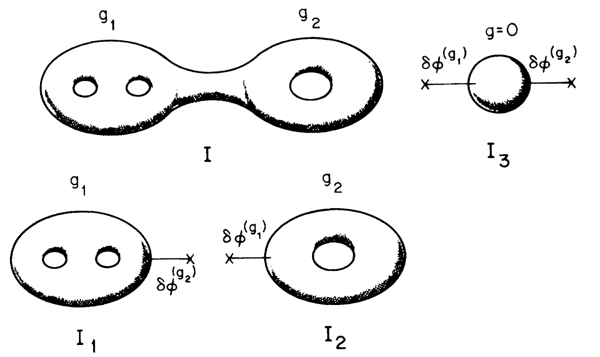

FIG. 1. Contributions needed to make the three-loop diagram well defined.

Let us build a family of pinching surfaces as follows. Choose a family of surfaces of genus g_1 with one marked point (X_1, P_1) depending on complex coordinates $m^1_{(1)}, \ldots, m^{3g_1-2}_{(1)}$. Repeat for side two. Then we get a family Δ of pinched surfaces parametrized by $\vec{m}_{(1)}, \vec{m}_{(2)}$ simply by gluing P_1 to P_2. To thicken these up, however, we need more data. For each $\vec{m}_{(1)}$ choose a local coordinate $z_{(1)}$ on X_1 centered at P_1, and similarly with side two. The usual "plumbing-fixture" construction then gives an almost pinched surface for each value of $\vec{m}_{(1)}, \vec{m}_{(2)}$, and a new variable q. The point we wish to make here is that while the construction depends on the slices $z_{(i)}$, $i = 1,2$, still it does so in a very special way. Suppose that we instead choose $z'_{(i)} = \alpha_{(i)}(\vec{m}_{(i)}) z_{(i)} + O(z^2_{(i)})$ for some functions $\alpha_{(i)}$. Then we get new coordinates; the surface formerly described by $\vec{m}_{(1)}, \vec{m}_{(2)}, q$, is now described by $\vec{m}'_{(1)}, \vec{m}'_{(2)}, q'$, where, in particular,

$$q' = \alpha_{(1)}(\vec{m}_{(1)}) \alpha_{(2)}(\vec{m}_{(2)}) q + O(q^2) . \tag{1}$$

Thus the logarithm of the leading coefficient is the sum of a term knowing nothing about side two, plus a term independent of side one. If we use only pinching coordinates q chosen in this way, then, we can hope that the prescription symbolized in Fig. 1 can cancel the "ambiguity" in the bosonic string integral. The details of how this works are given in Ref. 2.

Mathematically the above discussion can be succinctly restated as follows: Δ naturally has the structure of a Cartesian product, but moreover so does its *normal bundle* in the full moduli space. We can deal with tadpoles if we use transverse coordinates q which respect this extra structure. This is our proposal for how to generalize the coordinates used for the genus-2 zero-point function in Ref. 6 to arbitrary g and N.

When viewed in this way the heterotic case is very easy to understand. The integration measure itself is intrinsically defined on supermoduli space, a fact which is clearer in the coordinate-invariant operator formalism[12] than in the picture-changing language. The total derivative ambiguities appearing in coordinate-dependent formulas just describe indecision concerning the *region* of integration. As in the bosonic case there is no standard way to choose this cutoff region, but again the super-plumbing-fixture construction helps.[13,19] For a super (i.e., Neveu-Schwarz) pinch, one again finds a class of pinching coordinates related by (1), where now $\alpha_{(i)}$ depends on the even and odd moduli of side i. Such coordinates define a *superconformally covariant* cutoff. We must cut off the genus g_1+g_2 integration using such a q and then try to find insertions $\delta\phi_{\rm BG}$ on lower genus surfaces which cancel the dependence on chosen local superconformal coordinates $\mathbf{z}_1,\mathbf{z}_2$ near the attachment points. We need not consider spin (i.e., Ramond) pinches, since in the heterotic string Ramond states cannot disappear into the vacuum.

We should also be requiring that Becch-Rouet-Stora-Tyutin- (BRST) exact states decouple from loop amplitudes; indeed this condition subsumes the requirement that the dependence on $\mathbf{z}_1,\mathbf{z}_2$ cancel.[2,15] To implement it we first analyze the factorization of an amplitude with one exact state $Q\psi$ inserted. The sewing state is

$$\sum_a q^{L_0}\bar{q}^{\bar{L}_0} b_0^{(1)}\bar{b}_0^{(1)}\phi_a^{(1)}\otimes\phi^{a(2)}\,,$$

where the superscript denotes which Fock space $\phi^{(i)}$ lives in (or $b_0^{(i)}$ acts on), and $\phi^a=(\phi_a)^T$; the adjoint is defined by the standard sphere.[20] We then get

$$\int\frac{d^{10}k}{(2\pi)^{10}iK}q^{k^2/2-1/2}\bar{q}^{k^2/2-1}\exp\Big[\sum_{\mathbb{Z}_+}\{q^n(-\alpha_{-n}^{(1)}\cdot\alpha_{-n}^{(2)}+c_{-n}^{(1)}b_{-n}^{(2)}-b_{-n}^{(1)}c_{-n}^{(2)})+\text{conj.}\}$$
$$+\sum_{\mathbb{Z}_+-1/2}\{iq^k(\beta_{-k}^{(1)}\gamma_{-k}^{(2)}-\gamma_{-k}^{(1)}\beta_{-k}^{(2)}-d_{-k}^{(1)}\cdot d_{-k}^{(2)})-i\bar{q}^k\bar{d}_{-k}^{(1)}\cdot\bar{d}_{-k}^{(2)}\}\Big]$$
$$\times c_1^{(1)}\bar{c}_1^{(1)}c_1^{(2)}\bar{c}_1^{(2)}\delta(\gamma_{1/2}^{(1)})\delta(\gamma_{1/2}^{(2)})\,|k\rangle^{(1)}\otimes|k\rangle^{(2)}\,. \quad (2)$$

Here K is the string coupling defined in Ref. 2, d_k^μ are modes of the ten spacetime fermions, $\bar{d}_k^A$ are modes of the current-algebra fermions, and $\mathbb{Z}_+$ are the positive integers.

We can forget about the tachyons in (2); since G parity is not anomalous, any amplitude with all external states allowed by the modified Gliozzi-Scherk-Olive projection[14] will get no contribution from tachyons in separating pinches. (For the nonseparating case see Ref. 15.) Since the order of a differential at the divisor Δ is a coordinate-invariant number, it makes sense to say that the order of (2) is zero. Now multiply (2) by $|q|^{-2}dq\,d\bar{q}$ and consider changing coordinates from q to q'; then only the first Taylor coefficient of the change matters, and we see that we indeed have the situation discussed earlier. Among the leading (massless) terms of (2), only two are relevant, namely

$$\tfrac{1}{10}\,|\text{grav}\rangle\otimes|\text{grav}\rangle-\tfrac{1}{2}\,|\text{dil}\rangle\otimes|\text{dil}\rangle\,,$$

where

$$|\text{grav}\rangle=d_{-1/2}\cdot\bar{\alpha}_{-1}c_1\bar{c}_1\delta(\gamma_{1/2})\,|1\rangle$$

and

$$|\text{dil}\rangle=-(\tfrac{1}{2}c_1\gamma_{-1/2}+2c_1\bar{c}_1\bar{c}_{-1}\beta_{-1/2})\delta(\gamma_{1/2})\,|1\rangle\,.$$

Here $|1\rangle$ is the Osp(2|1)-invariant vacuum. Note that each of these states is physical, i.e., BRST closed. This is crucial to the success of our procedure; it is the additional condition on a heterotic theory mentioned in the introduction, a point also stressed in Ref. 9.

One might think that $|\text{dil}\rangle$ would decouple, being itself BRST exact. Instead it turns out to be the total derivative of a form which is not quite globally defined (see Refs. 11 and 15), and so can have a nonvanishing expectation value on a surface with a nontrivial Euler number $\chi=2-2g$. One gets

$$\langle\langle\text{grav}\rangle\rangle_g=-(i\pi/2)\times10\times(\chi-2)i\mathbb{Z}_g\,,$$
$$\langle\langle\text{dil}\rangle\rangle_g=-i\pi\chi i\mathbb{Z}_g\,, \quad (3)$$

where $i\mathbb{Z}_g\equiv\langle\langle1\rangle\rangle_g$ is the vacuum amplitude at g loops. The only effect of choosing the O(16)×O(16) string instead of the usual SO(32) case is that in the former case the g-loop partition function $i\mathbb{Z}_g\neq0$, while in the latter it vanishes.

As is well known, nonvanishing vacuum expectation values of massless fields at genus g_2 can lead to a failure of BRST invariance at genus g_1+g_2, when a BRST-exact state gives a total derivative which is nonzero on Δ. Accordingly we now introduce a BRST-*noninvariant* $\delta\phi_{BG}^{(g_2)}$ to eliminate the problem:

$$|\delta\phi_{\rm BG}^{(g_2)}\rangle=\frac{1}{4\pi}[(B_{\mu\nu}+h_{\mu\nu})d_{-1/2}^{\mu}\bar{\alpha}_{-1/2}^{\nu}c_1\bar{c}_1-\tilde{\Phi}(\tfrac{1}{2}c_1\gamma_{-1/2}+2c_1\bar{c}_1\bar{c}_{-1}\beta_{-1/2})$$
$$+(c_0+\bar{c}_0)(\zeta_\mu c_1 d_{-1/2}^{\mu}+\eta_\mu c_1\bar{c}_1\beta_{-1/2}\bar{\alpha}_{-1}^{\mu})]\delta(\gamma_{1/2})\,|1\rangle\,. \quad (4)$$

Here $B_{\mu\nu}$, $h_{\mu\nu}$, $\tilde{\Phi}$, ζ_μ, and η_μ are all functions of the zero mode of x^μ. Equation (4) is the most general background with quantum numbers appropriate to cancel the boundary term; for example, we have not included any gauge fields, as they will not be induced to lowest order in tadpoles. Applying the BRST operator to $|\delta\phi_{\rm BG}^{(g_2)}\rangle$ we get the condition for overall

BRST decoupling:

$$\tfrac{1}{2}\partial^2 h_{\mu\nu} - \tfrac{1}{2}\partial_\mu(\zeta_\nu + \tfrac{1}{2}\eta_\nu) - \tfrac{1}{2}\partial_\nu(\zeta_\mu + \tfrac{1}{2}\eta_\mu) = \eta_{\mu\nu}(8\pi^2/K)\tfrac{1}{10}\langle\langle \mathrm{grav}\rangle\rangle_{g_2}, \quad (5)$$

$$\tfrac{1}{2}\partial^2 B_{\mu\nu} + \tfrac{1}{2}\partial_\mu(\zeta_\nu - \tfrac{1}{2}\eta_\nu) - \tfrac{1}{2}\partial_\nu(\zeta_\mu - \tfrac{1}{2}\eta_\mu) = 0, \quad (6)$$

$$\tfrac{1}{2}\partial^\nu h_{\mu\nu} - \tfrac{1}{2}\partial^\nu B_{\mu\nu} + \tfrac{1}{2}\partial_\mu\tilde{\Phi} - \tfrac{1}{2}\eta_\mu = 0, \quad (7)$$

$$\partial^\nu h_{\mu\nu} + \partial^\nu B_{\mu\nu} + \partial_\mu\tilde{\Phi} - 2\zeta_\mu = 0, \quad (8)$$

$$\tfrac{1}{4}\partial^2\tilde{\Phi} + \tfrac{1}{2}\partial^\mu\zeta_\mu = (8\pi^2/K)\tfrac{1}{4}\langle\langle \mathrm{dil}\rangle\rangle_{g_2}, \quad (9)$$

$$\partial^2\tilde{\Phi} + \partial^\mu\eta_\mu = (8\pi^2/K)\langle\langle \mathrm{dil}\rangle\rangle_{g_2}. \quad (10)$$

After eliminating the auxiliary fields ζ and η, we get a homogeneous, decoupled equation for B. To make the remaining equations look more familiar we can define $\tilde{\Phi} = \Phi - \tfrac{1}{2}h^a_a$. Thus we have

$$\begin{aligned} &\partial^2\Phi = (8\pi^2/K)(\langle\langle \mathrm{grav}\rangle\rangle_{g_2} + \langle\langle \mathrm{dil}\rangle\rangle_{g_2}), \\ &\partial^\mu\partial^\nu h_{\mu\nu} - \partial^2 h^a_a + 2\partial^2\Phi = (8\pi^2/K)\langle\langle \mathrm{dil}\rangle\rangle_{g_2}. \end{aligned} \quad (11)$$

Using (3) we can now recognize (11) as the same equations of motion as those arising in the bosonic string,[2] namely the equations of linearized gravity plus dilaton with a cosmological constant given by the vacuum amplitude $i\mathbb{Z}$.

Thus we see that the loop-corrected equations of motion for physical background fields are well defined at g_1+g_2 loops if they are well defined at fewer loops. We therefore obtain inductively finite, unambiguous string dynamics, as in the bosonic case[21] and as in the no-tadpole case.[9]

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Note added.—We have recently learned that the result (1) has been worked out by Wolpert.[22]

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