General Relativity without the Metric

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A new class of generally covariant gauge theories is introduced. The only field in addition to the gauge connection is a scalar-density Lagrange multiplier. For the group SO(3,*C*) [SO(3,*R*)] in four dimensions and particular coupling constants, the theory is equivalent to complex [Euclidean] general relativity, modulo an important degeneracy. The spacetime metric is constructed from the curvature in a solution. A canonical analysis leads directly to Ashtekar's Hamiltonian formalism. The general solution to the four diffeomorphism constraints in the nondegenerate case is given.

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In the search for a more unified description of physics the analogy between general relativity (GR) and gauge theories has often been stretched. Gauge fields are described using connections whereas in GR the gravitational field is described in terms of the spacetime metric. While a spacetime connection always exists in GR, it is a derivative quantity of the metric, and the vacuum Einstein equation is really a condition on the metric, not on the connection. In this Letter we strengthen the analogy by expressing the vacuum Einstein equation in four spacetime dimensions purely in terms of the self-dual spin connection. The metric will play no role whatsoever, although in a solution it can be reconstructed from the curvature. As a byproduct we have found a new class of generally covariant gauge theories with local degrees of freedom.

Previous work on formulating GR as a gauge theory has fallen short of the goal in that the spacetime metric always entered as a fundamental field in one way or another. An early attempt based on a Palatini-type action involved both an SO(3,1) connection and an orthonormal tetrad (the "square root" of the metric) as fundamental variables.[1] Other work involved reinterpreting the tetrad as a connection gauging the group of spacetime translations.[2] While this is possible, the tetrad remains a basic variable and the equations of the theory are unchanged. More recently, a new Hamiltonian formulation of (3+1)-dimensional GR was discovered by Ashtekar in which the canonical coordinates are identical to those of an SO(3,*C*) Yang-Mills theory, with the self-dual spin connection conjugate to the orthonormal spatial triad.[3] A covariant Palatini-type action was found which leads directly to this new Hamiltonian formalism; however, just as in the earlier Palatini-type actions, the tetrad entered as an independent field.[4]

Here we complete the covariant gauge-theory formulation of vacuum GR by exhibiting a new action for complex GR which is a functional only of the self-dual spin connection and a scalar-density Lagrange multiplier field. It has the form of a constrained nonlinear σ model which is polynomial in the gauge potential and its first derivatives, and quadratic in "time" derivatives. *The spacetime metric does not appear in the action in any form.* We show (1) how the field equations which follow from this action reproduce the Einstein equation (modulo an important degeneracy), (2) how the spacetime metric can be built from the curvature in a solution, (3) how to impose conditions which select real Lorentzian or Euclidean solutions, (4) how Ashtekar's formulation of GR follows from the Hamiltonian analysis of this new action, and (5) how to construct the general solution of the four diffeomorphism constraints of the Ashtekar formalism in the nondegenerate case.

The discovery of a metric-free action for GR opens a new manifestly covariant line of attack on the problem of quantum gravity in concert with Ashtekar's canonical quantization program. It should be mentioned at the outset that flat spacetime does *not* naturally emerge from the theory at the classical level because an invertible Weyl curvature is needed to construct the spacetime metric in a solution. In fact, the classical perturbation theory about flat space does not even exist because the linearized dynamics is completely degenerate in that case. This degeneracy does not immediately contradict observation, however, and may, in any case, be irrelevant at the quantum level.

Other applications of these results remain largely to be seen. The fact that the Einstein equation can be written purely in terms of the left-handed spin connection may be of relevance in the search for a twistor theoretic construction of the general vacuum solution.[5] More generally, it may be useful in understanding the structure of the space of solutions. In another direction, this class of generally covariant gauge theories *other* than GR may be of interest in its own right. Moreover, the larger question of unification of gauge interactions *including gravity* might be usefully investigated in this framework.

Here is the action,

$$S[\eta,A^a]=\int h_{abcd}(\eta\cdot F^a\wedge F^b)F^c\wedge F^d\,, \tag{1}$$

with

$$h_{abcd}=\alpha(\delta_{ca}\delta_{bd}+\delta_{cb}\delta_{ad})+\beta\delta_{ab}\delta_{cd}\,. \tag{2}$$

The quantity η is a totally antisymmetric fourth-rank tensor, equivalent to a scalar density of weight -1, and the quantity A^a is an SO(3,C) connection with curvature $F^a:=dA^a+\frac{1}{2}\epsilon^a{}_{bc}A^b\wedge A^c$. Lower-case latin letters from the beginning of the alphabet range from 1 to 3 and are raised and lowered with the Kronecker δ. The "wedge" notation "$\wedge$" denotes antisymmetric outer product of forms, and the dot "$\cdot$" just indicates the contraction $\eta^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$. We shall also make use of the three- and four-dimensional Levi-Civita antisymmetric tensor densities $\epsilon^{\mu\nu\rho\sigma}$ and ϵ^{ijk}, which have components equal to ± 1 for even or odd permutations of the indices and vanish whenever two indices coincide.

General relativity corresponds to the choice of coupling constants $\alpha/\beta=-1$ for the two SO(3,C)-invariant "traces" in the action.[6] Other choices for the couplings or the group define other, generally covariant gauge theories. Generalizations of (1) to any spacetime dimension d are obtained by contracting d curvatures with two Levi-Civita ϵ tensors and a constant rank-d group-invariant tensor, and multiplying by a scalar density of weight -1. The action remains second order in "time" derivatives due to the antisymmetry of the ϵ's. We do not know of a modification of (1) corresponding to GR with a cosmological constant.

One way to see that (1) is indeed an action for GR in the nondegenerate case is provided by the equivalence of its Hamiltonian formulation with that of Ref. 3. Alternatively, in a covariant approach, we first reexpress the vacuum Einstein equation in terms of a set of three (complex) spacetime two-forms Σ^a, an SO(3,C) connection A^a, and a symmetric traceless tensor ψ_{ab} as

$$\Sigma^a\wedge\Sigma^b=\tfrac{1}{3}\delta^{ab}\Sigma^c\wedge\Sigma_c\,, \tag{3a}$$

$$D\Sigma^a=0\,, \tag{3b}$$

$$F^a=\psi^a{}_b\Sigma^b\,, \tag{3c}$$

where D denotes the covariant exterior derivative with respect to A^a, $D\Sigma^a:=d\Sigma^a+\epsilon^a{}_{bc}A^b\wedge\Sigma^c$. The constraint (3a) says that the tracefree part of $\Sigma^a\wedge\Sigma^b$ vanishes, which implies[7] the existence of a set of four (complex) one-forms θ^a such that Σ^a is determined by the (Minkowski) self-dual part of $\theta^\alpha\wedge\theta^\beta$, i.e.,

$$\Sigma^a=\theta^0\wedge\theta^a-i\tfrac{1}{2}\epsilon^a{}_{bc}\theta^b\wedge\theta^c\,. \tag{4}$$

When (3) holds with $\Sigma^c\wedge\Sigma_c\neq 0$, the spacetime metric $g_{\mu\nu}:=\theta^\alpha_\mu\theta^\beta_\nu\eta_{\alpha\beta}$ is Ricci flat, i.e., it satisfies the vacuum Einstein equation.[8]

To better understand the role of Eq. (3a), note that if the tracefree part of $\Sigma^a\wedge\Sigma^b$ were nonzero, it would determine at least one eigenvector, thus breaking the SO(3,C) symmetry. Since a spacetime metric does not break this symmetry, (3a) is necessary if Σ^a is to be derivable from a tetrad θ^a. The five conditions (3a) on the 18 degrees of freedom in Σ^a leave 13 degrees of freedom, which is equal to the 16 degrees of freedom in θ^a less the three dimensions of anti-self-dual Lorentz transformations under which (4) is invariant.

The tetrad θ^a determines a metric compatible spin connection $\omega_\beta{}^\alpha$ via the torsion-free condition $d\theta^\alpha+\omega^\alpha{}_\beta\wedge\theta^\beta=0$. Equation (3b) implies that, when $\Sigma^c\wedge\Sigma_c\neq 0$, A^a is determined by the self-dual part of $\omega^\alpha{}_\beta$, i.e.,

$$A^a=-i\omega^{0a}+\tfrac{1}{2}\epsilon^a{}_{bc}\omega^{bc}\,. \tag{5}$$

In the presence of (3a) and (3b), (3c) says that the only nonvanishing part of the self-dual projection of the curvature of $\omega^\alpha{}_\beta$ is the pure spin-2 (Weyl) part, so that the metric $g_{\mu\nu}$ must be Ricci flat.[9]

Varying η and A^a in the action (1) we obtain the equations of motion

$$h_{abcd}(\epsilon\cdot F^a\wedge F^b)F^c\wedge F^d=0\,, \tag{6a}$$

$$D[h_{abcd}(\eta\cdot F^a\wedge F^b)F^c]=0\,. \tag{6b}$$

With the definitions

$$\Sigma_d:=h_{abcd}(\eta\cdot F^a\wedge F^b)F^c\,, \tag{7a}$$

$$\psi^a{}_b:=\{[h(\eta\cdot F\wedge F)]^{-1}\}^a{}_b\,, \tag{7b}$$

the field equations (6) imply the Einstein equations (3) *provided* the coupling constants have the ratio $\alpha/\beta=-1$. Indeed, (3b) follows from (6b), (3c) is true by definition, and it is straightforward to show using (6a) and $\alpha/\beta=-1$ that $\psi^a{}_b$ is tracefree and (3a) is also true. (The characteristic equation satisfied by any 3×3 matrix is useful in showing these last two steps.)

Summarizing our results up to this point, given a connection A^a and a totally antisymmetric tensor η satisfying (6a) and (6b), the tetrad determined via (4) by Σ^a of Eq. (7a) is a vacuum solution of Einstein's equation. Conversely, every vacuum solution with $\det\psi\neq 0$ arises in this manner from A^a given by (5) together with $\eta^{\mu\nu\rho\sigma}=i(2\alpha\det\psi)^{-1}\theta^{\mu\nu\rho\sigma}$, where $\theta^{\mu\nu\rho\sigma}$ is the inverse volume element. These results are local in nature; if the manifold is not orientable or does not admit a spin structure, there can be obstructions to their global form. Note that within each diffeomorphism class of solutions one can find a solution where, at least locally, η has any prescribed value. Using this freedom to fix η, GR is thus expressed purely in terms of the self-dual spin connection A^a.

The equivalence between the sets of Eqs. (3) and (6) breaks down when $\det[h(\eta\cdot F\wedge F)]=0$ or $\det\psi=0$. That is, solutions to (3) with $\det\psi=0$ do not arise via (7) from solutions to (6). For example, ordinary flat space with a nondegenerate Σ^a is in this category. More generally, according to the algebraic classification of the Weyl tensor, $\det\psi=0$ if and only if (i) the Weyl tensor is of type {1111} and there exists a Lorentz frame in which the gravitational principal null directions lie at the vertices of a square, or (ii) the Weyl tensor is of type {31}, {4}, or {−};[10] there are no vacuum solutions for the case (i), however.[11] On the other hand, solutions to (6) with $\det[h(\eta\cdot F\wedge F)]=0$ do not arise from solutions to (3). For example, all "static" configurations ($A_0^a=0$, $\partial_0 A_i^a=0$, $\eta=$anything) are solutions to (6), but do not arise from solutions to (3), since $\Sigma^a=0$ and $\psi_{ab}=\infty$.

To select real Lorentzian solutions, one must impose the conditions

$$\Sigma^a\wedge\bar{\Sigma}^b=0,\quad \mathrm{Re}(\Sigma^a\wedge\Sigma_a)=0\,, \tag{8}$$

which imply that θ^a can be chosen real in (4) (up to a possible irrelevant overall eighth root of unity). Given the definition of Σ^a in (7a) these become sixth-order algebraic conditions on the curvature F^a. (When $\psi^a{}_b$ is nondegenerate the first condition is equivalent to the second-order condition $F^a\wedge\bar{F}^b=0$.) Real Euclidean solutions are selected by requiring simply that A^a and η be real, since then Σ^a of (7a) is real and there exists a real tetrad θ^a satisfying (4) with the i replaced by either 1 or -1 in the second term.

The Hamiltonian formulation of the theory is derived from a 3+1 decomposition of the action (1) with respect to coordinates (x^0,x^i), treating x^0 as the time coordinate. First, we simply rewrite (1) as

$$S=\int \tfrac{1}{2}G_{ab}^{ij}(\eta,A_i)(\dot{A}_i^a-D_iA_0^a)(\dot{A}_j^b-D_jA_0^b)\,,$$

with

$$G_{ab}^{ij}(\eta,A_i):=\eta h_{acbd}B^{ic}B^{jd}\,, \tag{9}$$

$$B^{ic}:=\epsilon^{ijk}F_{jk}^c\,, \tag{10}$$

where η is now thought of as a scalar density of weight -1. The action thus takes the form of a nonlinear σ model, with degeneracies arising from diffeomorphism and SO(3,C) invariances. More precisely, let the canonical momenta be defined in the usual way by $\pi_a^\mu:=\delta L/\delta\dot{A}_\mu^a$, $\pi_\eta:=\delta L/\delta\dot{\eta}$. Then one has

$$\pi_a^i=G_{ab}^{ij}F_{0j}^b,\quad \pi_a^0=0,\quad \pi_\eta=0\,. \tag{11}$$

Since G_{ab}^{ij} is degenerate, the "velocities" $\dot{A}_i^a$ are not uniquely determined by the momenta π_a^i, and there are three corresponding constraints,

$$\epsilon_{ijk}\pi_a^jB^{ka}=0\,. \tag{12a}$$

The vanishing of π_a^0 gives the Gauss-law constraint,

$$D_i\pi_a^i=0\,, \tag{12b}$$

in the usual way as a secondary constraint. Finally, assuming the "magnetic" field (10) to be nondegenerate, the vanishing of π_η gives rise to a secondary constraint which, modulo a combination of (12a) and (12b), takes the form

$$\epsilon_{abc}\epsilon_{ijk}\pi^{ia}\pi^{jb}B^{kc}=0\,. \tag{12c}$$

The constraints (12a), (12b), and (12c) are precisely those found by Ashtekar via a canonical transformation in the usual phase space of GR.[3] The Hamiltonian is an arbitrary linear combination of these constraints (up to a boundary term) and evolution is generated by the usual Hamiltonian equations. When the magnetic field (10) is degenerate, the "field space metric" G_{ab}^{ij} has additional degeneracies and the evolution is correspondingly underdetermined.

The initial-value constraints of the Hamiltonian formalism are merely those field equations which do not involve second time derivatives. Since the momentum π_a^i is related to the spatial component of the two-form defined in (7a) by $\pi_a^i=\epsilon^{ijk}\Sigma_{jka}$, we have by inverting (3c) that

$$\pi_a^i=(\psi^{-1})_{ab}B^{ib}=[\phi^2-\tfrac{1}{2}\,\mathrm{tr}\phi^2]_a{}^bB_b^i\,, \tag{13}$$

where $\phi_{ab}:=(\det\psi)^{-1/2}\psi_{ab}$, and the tracelessness of ψ has been used in the second equality. Conversely, given any connection A_i^a, the diffeomorphism constraints (12a) and (12c) are identically satisfied when π_a^i has the form (13), with ϕ_{ab} tracefree. When B^{ia} is nondegenerate, this five-parameter family is the general solution to these constraints. All that remain are the Gauss-law constraints, which now take the form

$$B^{ia}D_i[\phi^2-\tfrac{1}{2}\,\mathrm{tr}\phi^2]_{ab}=0\,, \tag{14}$$

where the Bianchi identity $D_iB^{ia}=0$ has been used. The two-parameter family of tracefree solutions ϕ_{ab} to (14) represent the unconstrained degrees of freedom in the momenta conjugate to the connection. This solution of the diffeomorphism constraints generalizes the solution $\pi_a^i=\Lambda^{-1}B_a^i$ for the case of self-dual solutions with cosmological constant found in Ref. 12.

We did not discover the action (1) by virtue of its uniqueness, although in retrospect that would have been possible. Rather we began with a new action for GR,

$$S[\Sigma^a,A^a,\psi_{ab}]=\int \Sigma^a\wedge F_a-\tfrac{1}{2}\,\psi_{ab}\Sigma^a\wedge\Sigma^b\,,$$

and proceeded to eliminate both Σ^a and ψ_{ab} via their equations of motion. This alternate formulation allows one to incorporate couplings to gauge fields, fermions, and the gravitino field of supergravity.[7] The results of Ref. 7 may be useful for the inclusion of matter couplings in the pure connection formulation presented in this Letter.

This project grew out of a remark made by Lionel Mason to the effect that the self-dual two-form Σ^a can be employed as the basic variable in general relativity.[13]

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[6]In two-component spinor notation, the action (1) with $\alpha/\beta=-1$ becomes $S[\eta,A^{AB}]=\int(\eta\cdot F^{AM}\wedge F^{BN})F_{AB}\wedge F_{MN}$, where F^{AB} is the curvature two-form of the SL(2,C) connection A^{AB}, and indices are raised and lowered with the antisymmetric spinor ϵ_{MN} and its inverse.

[7]R. Capovilla, J. Dell, T. Jacobson, and L. Mason (to be published).

[8]If the Euclidean metric $\delta_{\alpha\beta}$ is used instead, with i being replaced by 1 in the second term of (4), a new tetrad θ'^{α} is obtained, but the resulting (complex) "Riemannian" metric $g'_{\mu\nu}:=\theta'^{\alpha}_{\mu}\theta'^{\beta}_{\nu}\delta_{\alpha\beta}$ just i times the previous "Lorentzian" one.

[9]For a spinor version of this statement, see, e.g., R. Penrose and W. Rindler, *Spinors and Spacetime* (Cambridge Univ. Press, Cambridge, 1984), Vol. 1.

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