Superconductive Pairing of Fermions and Semions in Two Dimensions

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We have observed, in exact numerical solutions of small systems, the microscopic precursors of superconductive pairing of fermions and semions (half-statistics quasiparticles) in two dimensions. We recognize the paired state by flux quantization at intervals of $hc/2e$. We find that the fermions pair only for values of an interparticle potential u which is large and negative, while the semions pair for a wide range of u, including strong repulsion. We also find that the semions, in the paired state, prefer quantized flux in *odd* multiples of $hc/4e$.

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The current interest in two-dimensional superconductivity has led to claims that, first, the ground state (GS) of a gas of particles of rational fractional statistics (anyons)[1,2] is a superfluid[3-8] for which the bosonic degrees of freedom are composites of an integral number of anyons; and that, second, the high-temperature superconductors[3] are an experimental realization of this possibility, the anyons in this case being half-statistics objects (semions), which pair—as do fermions—in the superfluid state (and thus are expected to show flux quantization at multiples of $hc/2q$, where q is their charge)[3,4,9,10] Each of these ideas is interesting in itself. Clarification of the existence and nature of anyon superconductivity is an attractive and challenging problem for theorists; it is also a necessary prerequisite to experimental tests which may distinguish semion from fermion superconductivity in the high-temperature superconductors. In the following, we address ourselves to these questions. Besides fermions, we have concentrated on semions because (i) they are expected to form the most robust of the purported anyon superfluids, the composites being simply pairs; and (ii) they are the case of interest for the copper oxides.

Most methods which are used in quantum mechanics for the many-particle problem do not readily carry over for objects with fractional statistics. We have, however, found that it is straightforward to study small numbers of anyons on a finite lattice by exact diagonalization.[11] We have used a rectangular lattice, with periodic boundary conditions in the x direction only. Our lattice is thus topologically equivalent to a cylinder, or, alternatively, to a punctured disk, with the lattice sites on "spokes" of the disk.

For a charged superfluid (from now on we take the single-particle charge to be e), we expect the GS energy to show evenly spaced minima as a function of test flux inserted through the hole of the cylinder,[12] with spacing $hc/ne \equiv \phi_0/n$, where n is the number of particles forming the composite—that is, 2, for fermions and semions. We take our Hamiltonian to be

$$H = T + V = -t\sum_{\langle ij\rangle}[c_i^\dagger c_j \exp(i\phi_{ij}) + \text{H.c.}] + u\sum_{\langle ij\rangle} n_i n_j\,, \quad (1)$$

where $\langle ij\rangle$ refers to near-neighbor sites, and

$$\phi_{ij} = \frac{e}{\hbar c}\int_i^j [\mathbf{A}_s + \mathbf{A}_c]\cdot d\mathbf{l}\,.$$

The phase upon hopping arises from both the other anyons ($\mathbf{A}_s$) and the test flux ($\mathbf{A}_c$). In this representation, the anyons are hard-core bosons pierced by flux tubes.[1,11] Fermions are then simply anyons[13] whose phase shift upon exchange is $\exp(\pm i\pi)$. We have taken $t=1$, and explored a wide range of u.

The obvious problem with small-system calculations of flux quantization (FQ) is that any small system will show some structure in GS energy versus flux—for instance, one-dimensional particles always show perfect FQ.[12] We have thus restricted ourselves to systems large enough that they do not trivially show FQ. In practice, this has meant systems with 15 sites (3×5, where the second number is the number of sites in the x or azimuthal direction), 16 sites (4×4), and 20 sites (4×5), with ≥ 6 particles (or holes). For compactness we designate 16 sites with 8 particles as $4\times4/8$, etc.

We consider first the fermions as a test of the feasibility of our approach. Within the above restrictions (and an upper bound on the Hilbert space due to computational limitations), we have found three systems of fermions which show nontrivial FQ appropriate to paired states.[14] One of the three is illustrated in the middle curve of Fig. 1, which shows E_{GS} vs $\alpha_c \equiv \phi/\phi_0$ (where ϕ is the test flux through the hole) for fermions on the 4×4 lattice. The center curve shows FQ[15] at multiples of $\phi_0/2$, with a near-neighbor interaction $u=-2$, for 8 fermions. We identify this state as the microscopic precursor of the thermodynamic, superconducting state.

The upper curve of Fig. 1 demonstrates the absence of pairing for free fermions. Its several minima are expect-

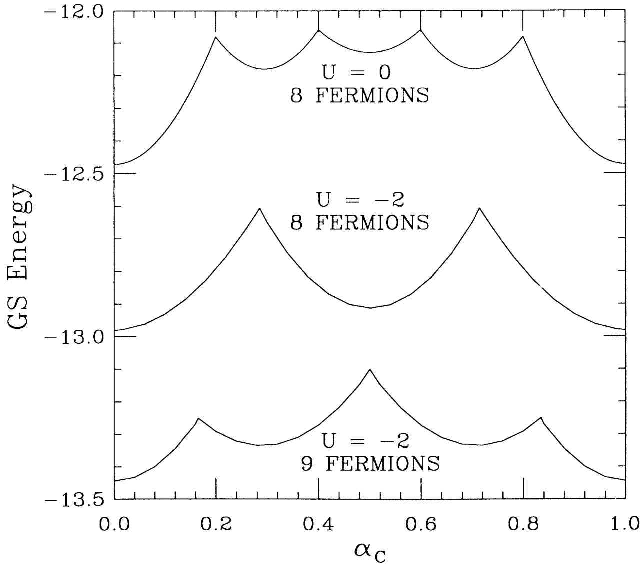

FIG. 1. Ground-state energy E_{GS} for fermions on the 4×4 lattice, as a function of test flux $\alpha_c = \phi/\phi_0$. Two of the curves are displaced in energy so as to lie in the same plot. Only the middle curve shows flux quantization appropriate to a paired state.

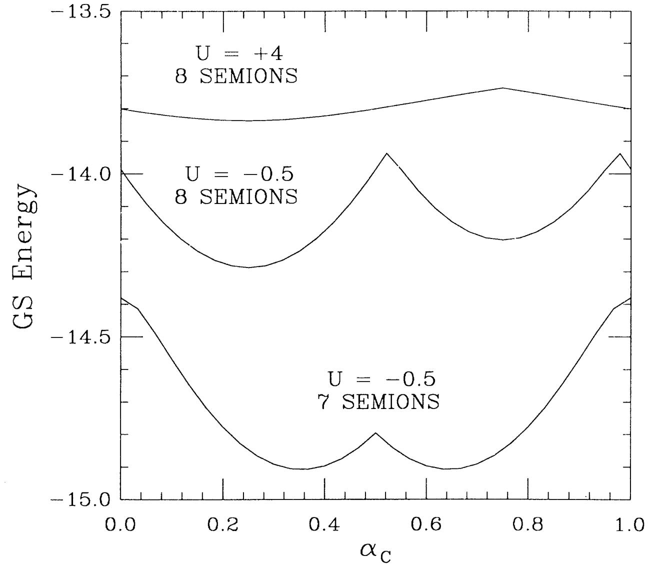

FIG. 2. Same as Fig. 1, except the particles are semions. Note that the energy minima in the paired state (middle curve) are displaced in test flux from the fermion case by $\phi_0/4$, and that the curves for 8 semions are not symmetric about zero flux.

ed to evolve, with increasing system size and particle number n_p into many closely spaced minima of roughly equal energy, giving in the thermodynamic limit a curve of E_{GS} versus test flux that is flat—i.e., no FQ. The lower curve (9 fermions) shows a finite-size effect (the effect of odd particle number) which nevertheless supports our conclusion that the middle curve shows pairing. We have examined many cases with odd n_p but failed to find any showing the appropriate FQ.

In one sense Fig. 1 is deceptive. The states which constitute the GS for various ϕ at $u=0$ (upper curve) do not smoothly evolve into the two states (identified with the two minima[12,16]) seen in the middle curve, as $u \to -2$. Each of the two pair states, in fact, appears as a level crossing at some u_1, and is thus orthogonal to the GS for $u > u_1$. This is readily seen by plotting $\langle V \rangle$ and $\langle T \rangle$ as functions of u; at the pairing transition, each is discontinuous—with $\langle V \rangle$ falling and $\langle T \rangle$ rising—as would be expected if pairing is occurring.

Besides 4×4/8, we have seen pairing of fermions for the 4×4/10 and the 4×5/14 systems (each thus with 6 holes). We have not seen pairing for 6 fermion particles for any system. (This is another finite-size effect: Particle-hole symmetry is not exact for fermions in a system with hard walls.) With so few cases we cannot make strong statements about the size dependence of the FQ; one should see the energy barrier (EB) separating the two states scaling as n_p. Our results are consistent with this scaling; however, the EB for the three cases is 0.310 (8 particles), 0.239 (6 holes), and 0.225 (6 holes)—all taken at $u=-2$.

We note also that our fermions are spinless, and thus presumably exhibit p-wave pairing. All three cases (see Fig. 3) show the strongest FQ at $u \sim -2$. We find it of interest to examine this result (extrapolated to the thermodynamic limit) in the light of the results of Randeria, Duan, and Shieh (RDS),[17] who show for a continuum model that the existence of a two-particle bound state is a necessary condition for a many-particle s-wave instability, but not for higher angular momentum. We have found analytically that two spinless fermions on an infinite square lattice have a bound state for $u < u_c = 2\pi/(2-\pi) \simeq -5.5$. This is consistent with our numerical results for two particles in finite systems, which show a strong rise in $\langle V \rangle / u$ vs $-u$ at $u \sim -5$. We thus find, in agreement with the result of RDS, a non-s-wave many-particle paired state in two dimensions, for u above the threshold u_c for a two-body bound state (i.e., $|u| < |u_c|$).[18]

We turn now to the results for semions. Again excluding smaller systems (and one marginal case of 6 particles on 15 sites), we have found four cases exhibiting FQ appropriate to the paired state. An example (3×5/8) is shown in Fig. 2. As in Fig. 1, we show the paired state at $u=u_p$ (middle curve), a state for larger u which is not paired, and a state with odd n_p at $u=u_p$. We concentrate on the differences from the fermion case. The obvious difference is that (as conjectured by Zhang[19]) E_{GS} of the paired state is *not* symmetric around $\phi=0$. The pair state shows FQ (energy minima) at values

$$\phi_Q = (hc/e)(\ldots, -\tfrac{3}{4}, -\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{3}{4}, \ldots) . \qquad (2)$$

We find, in fact, that the symmetry point in flux is always displaced from zero by an amount

$$\bar{\phi}/\phi_0 = -(n_p - 1)\alpha_s/2\,(\mathrm{mod}\,1)\,, \tag{3}$$

where α_s is the statistics ($\frac{1}{2}$ for semions). We can perhaps explain this result with a simple argument. Imagine the disk deformed to a cylinder, and picture the anyons as bosons with attached flux tubes which penetrate the cylinder walls. Then a choice of gauge for the flux tubes amounts to a choice of *how* the flux tubes exit from the cylinder—some out the top, some out the bottom—which may be ascertained by letting one of the anyons orbit either opening. Clearly, if we let half the anyon flux out the top, and half out the bottom, the resulting system is, by its symmetry, insensitive to the sign of any *test* flux threaded through the hole. Now deform this system back to a disk. The disk now has a nonzero flux through its (single) hole; it also has the problem that nontrivial exchange loops (around the hole) do not give the correct phase. One readily finds that both of these problems are cured by inserting $\bar{\phi}$ through the center, which amounts to correcting the gauge choice for the disk. Thus the correct gauge choice (that with zero test flux through the hole) differs from one which is symmetric about zero test flux by $\bar{\phi}$.

This result may have experimental implications. Imagine a macroscopic system of many sheets of superconducting semions, punctured by a hole. Then, assuming the superconducting state to be a superposition of states of *even* n_p, the system should show FQ at values given by Eq. (2) above, in clear distinction to a fermion superconductor. Furthermore, as noted by Yang,[12] the Meissner effect is intimately related to flux quantization. Thus an anyon superconductor which "prefers" nonzero flux through all closed loops which are confined to the superconducting region (e.g., lattice plaquettes) may show a very different response to applied magnetic fields from that of a T-invariant, fermion superconductor. An added complexity arises from the fact that anyons are expected to appear in real systems as vortices in a background fluid, in which case they are invariably subject to an effective magnetic field due to the background.[20,21] This latter question has, to our knowledge, not been addressed in treatments of anyon superfluidity; it may prove important in clarifying the physical conditions under which anyon superconductivity may be realized.

We summarize some of our observations on semion and fermion pairing in Fig. 3. The range of values of u for which pairing (by our FQ criterion) was observed is shown by an "error bar," while that value u_p at which the pairing is strongest (as measured by EB) is marked by a symbol, for each case. The fermion cases were always bounded by a level crossing in u, both above and below. In contrast, for some semion cases, the EB simply became very small as u deviated strongly from u_p; these cases are marked with dashed lines. We note that

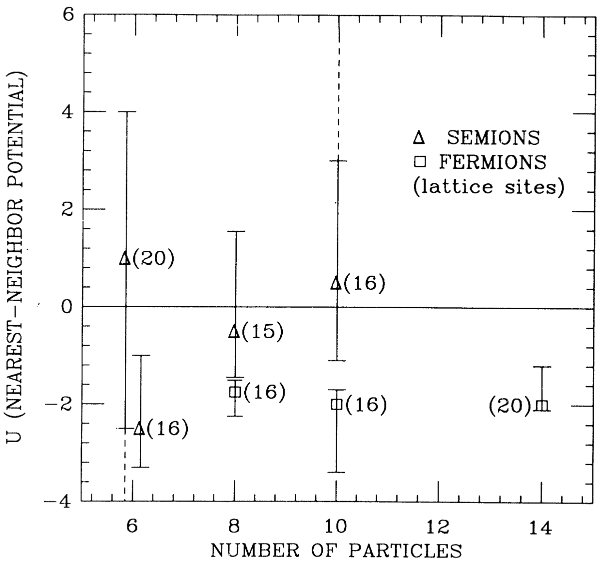

FIG. 3. Cases for which we have found paired states. The range of the nearest-neighbor potential u is indicated with error bars; points of strongest pairing are indicated with squares (fermions) and triangles (semions). The number of lattice sites is given in parentheses beside each symbol. No pair states were observed for odd particle number.

the larger (semion) systems show a larger range in u, and extend well into the range of positive u, as predicted by Laughlin.[4] There is thus no question of a two-body bound state for the semions. There were also two systems—$4\times5/14$, and $4\times4/8$—where we might have expected to see semion pairing and did not. The former case showed a behavior of E_{GS} versus test flux identical to that of several fermion cases which "almost" showed pairing. The latter case ($4\times4/8$) is, however, not understood. We can only speculate that some lattice commensuration effect[22] dominates the physics in this case. The size (n_p) dependence of the EB at u_p was as follows: 0.184 ($4\times4/6$), 0.224 ($4\times5/6$), 0.276 ($3\times5/8$), and 0.256 ($4\times4/10$). Thus, EB again roughly scales with system size and with n_p (up to half filling); but further work with larger systems is clearly needed to substantiate these limited data. We remark finally that we have searched, without success, for signs of triplets of $(2\pi/3)$-statistics particles (which should give FQ with spacing $hc/3e$)[3,6] in systems with 9 particles; we assume our lack of success is due to the small size of our systems.

In summary, we have observed paired states—the microscopic analogs of macroscopic superconductivity—in small systems of fermions and half-statistics particles, or semions, on a lattice. We find that the fermions pair only for values of the interparticle potential u which are large and negative, but still above the threshold for a two-particle bound state. In contrast, the semions pair for a wide range of u, including large positive u. We suggest a possible experimental signature to distinguish semion pairing—i.e., flux quantization at *odd* multiples of $hc/4e$. We plan to extend this work to larger systems, and to examine the magnetic field dependence of the paired states, in future work.

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[1]Frank Wilczek, Phys. Rev. Lett. **49**, 957 (1982); Phys. Rev. Lett. **48**, 1144 (1982); R. Mackenzie and Frank Wilczek, Int. J. Mod. Phys. A **12**, 2827 (1988).

[2]Y. S. Wu, Phys. Rev. Lett. **52**, 2103 (1984); Phys. Rev. Lett. **53**, 111 (1984).

[3]R. B. Laughlin, Science **242**, 525 (1988).

[4]R. B. Laughlin, Phys. Rev. Lett. **60**, 2677 (1988).

[5]A. L. Fetter, C. B. Hanna, and R. B. Laughlin, Phys. Rev. B **39**, 9679 (1989).

[6]J. K. Jain and N. Read, Phys. Rev. B **40**, 2723 (1989).

[7]Y.-H. Chen, F. Wilczek, E. Witten, and B. Halperin, Int. J. Mod. Phys. B **3**, 1001 (1989).

[8]D.-H. Lee and M. P. A. Fisher, Phys. Rev. Lett. **63**, 903 (1989).

[9]V. Kalmeyer and R. B. Laughlin, Phys. Rev. Lett. **59**, 2095 (1987).

[10]X. G. Wen, Frank Wilczek, and A. Zee, Phys. Rev. B **39**, 11413 (1989); X. G. Wen and A. Zee, Institute for Theoretical Physics Report No. NSF-ITP-88-150, 1988 (to be published).

[11]G. S. Canright, S. M. Girvin, and A. Brass, preceding Letter, Phys. Rev. Lett. **63**, 2291 (1989).

[12]C. N. Yang, Rev. Mod. Phys. **34**, 694 (1962); N. Byers and C. N. Yang, Phys. Rev. Lett. **7**, 46 (1961).

[13]This representation of fermions is exact; we can check the correspondence with exact analytical solutions (for the noninteracting case) with arbitrary test flux.

[14]We attempt to reserve the term "superconducting" for the thermodynamic limit of the paired state.

[15]It is important not to be misled by the schematic nature of the relevant figures in Ref. 12. A careful reading of these papers, and of J. R. Schrieffer, *Theory of Superconductivity* (Benjamin, New York, 1964), pp. 240–244, shows that (a) the energies of the *adjacent* minima are not necessarily equal, and (b) the maxima are, in fact, level crossings between orthogonal states, and are thus cusps rather than the smooth curves shown in Ref. 12. In the thermodynamic limit, the energy difference should remain of $O(1)$, and the cusps should persist.

[16]Schrieffer, Ref. 15.

[17]M. Randeria, J.-M. Duan, and L.-Y. Shieh, Phys. Rev. Lett. **62**, 981 (1989).

[18]The s-wave result of RDS may not hold for fermions on a lattice, since their model assumes a constant density of states (Ref. 17).

[19]S. C. Zhang, Institute for Theoretical Physics Report No. NSF-ITP-88-188i, 1988 (to be published).

[20]F. D. M. Haldane, Phys. Rev. Lett. **51**, 605 (1983).

[21]Daniel Arovas, J. R. Schrieffer, and Frank Wilczek, Phys. Rev. Lett. **53**, 722 (1984).

[22]E. Fradkin (private communication) has suggested that the half-filled lattice for even statistics is indeed a special case.