# Statistics and Flux in Two Dimensions

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(Received 10 April 1989)

We have investigated the interplay of fractional statistics and magnetic field for anyons on a lattice, by exact calculation of ground-state energies and wave functions for small systems. We find that the two types of "flux" (magnetic and statistical) are to some extent interchangeable in determining the ground-state properties, and thus that mean-field theories, which replace statistical phase shifts by magnetic fields, may be an appropriate method for treating fractional statistics.

PACS numbers: 74.65.+n, 05.30.−d, 67.90.+z, 73.40.−c

There is currently considerable interest in the properties of objects with fractional statistics (anyons). Identical anyons give a phase upon exchange which is neither $+1$ (Bose) nor $-1$ (Fermi), but rather a complex phase; such objects can only be defined consistently in two dimensions.[1] The name "anyon" was coined by Wilczek,[2] who proposed a model for the anyon as a bound composite of a flux tube and a charged particle. There is considerable theoretical evidence[3,4] that the quasiparticles (vortices) in the fractional quantum Hall effect[5] may be characterized by fractional statistics. Recently, Kalmeyer and Laughlin[6] and others[7] have suggested that the quasiparticles of frustrated two-dimensional magnets also have fractional statistics, and further that half-statistics objects (semions[8]) can pair to form a superfluid state, which may in fact be important for understanding the high-temperature superconductors.[6,7,9]

Calculations with anyons are difficult since their wave functions are neither symmetric nor antisymmetric.[10] They may, however, be viewed as ordinary fermions (or hard-core bosons) with an attached flux tube, giving a modified Hamiltonian with a long-ranged interaction which yields the appropriate phase change upon exchange. In this picture one can imagine a mean-field theory of anyons[6,10,11] in which one particle moves in the smeared-out, average flux field of the other, i.e., in a uniform magnetic field.[12] We have calculated numerically the exact ground-state properties of small systems of anyons [including fermions and hard-core (HC) bosons] on a lattice, in a uniform magnetic field. Our aim has been to test the applicability of the mean-field theory (MFT) of anyons, by examining the interplay of statistics (attached flux) and magnetic field ("detached statistics") in two dimensions. We view our anyons as charged, HC bosons with attached flux tubes. The strength of the flux tube (the statistics) is given by the parameter $\alpha_s$, where the phase due to exchange is $\theta_s = \pi\alpha_s$. The magnetic field is given by $\alpha_{\rm mf}$, the number of flux quanta per plaquette. Our Hamiltonian may thus be written as

$$H = -t\sum_{\langle ij\rangle}[c_i^\dagger c_j \exp(\phi_{ij}) + \text{H.c.}]\,,$$

where $\phi_{ij}$ is the phase due to other anyons, and to the magnetic field, arising from the hop $ij$. We take $t=1$. Since wave-function overlaps (see below) are gauge dependent, we have applied the same gauge choice to both the flux attached to the anyons, and the plaquette flux. We have examined rectangular lattices with periodic boundary conditions in one ($x$) direction. We have block diagonalized $H$, finding the lowest-energy state for each wave vector $k_x$, using a modified Lanczos technique.[13]

To the extent that MFT is correct, the properties of a system of anyons depend only on a weighted sum of the "statistical" flux and the background flux, given by

$$\tilde{\alpha}_s = \alpha_s + \alpha_{\rm mf}/n_a\lambda\,, \tag{1}$$

where $n_a$ is the lattice filling and $\lambda$ is a finite-size correction—the ratio of lattice sites to plaquettes. We have taken $\lambda = 1$.

Figure 1 shows the ground-state (GS) energies as a function of $\tilde{\alpha}_s$ for various combinations of $\alpha_s$ and $\alpha_{\rm mf}$. The lattice is "3×4/6" in our notation, which means 3 rows ($y$ direction) and 4 columns ($x$ direction), with 6 particles ($n_a = \frac{1}{2}$). We see that the GS energies are, to a good approximation ($\sim 5\%$), a function only of $\tilde{\alpha}_s$; i.e., that MFT is not a bad predictor of the GS energy. We have plotted similar data for three other lattices: 3×4/8, 4×4/6, and 4×4/8. We find that the good fit seen in Fig. 1 becomes less good as we deviate from half filling; i.e., the spread in GS energies is about 10%, 13%, and 6%, respectively, for these three cases (the last being again half-filled).

One can also test whether the exact GS of an anyon system has a significant overlap with the GS of the corresponding MFT state in a Bose or Fermi representation. Let $|\alpha_s, \alpha_{\rm mf}\rangle$ be the GS for a system with the indicated

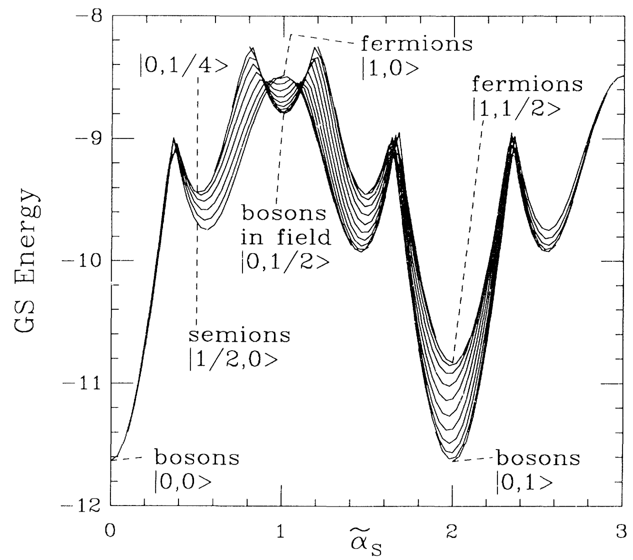


FIG. 1. Ground-state energies on the 3×4/6 lattice vs "effective statistics" $\tilde{\alpha}_s$ (see text) for various combinations $|\alpha_s,\alpha_{mf}\rangle$ (a few are marked). Each curve is a sweep of real statistics $\alpha_s$ at a different magnetic field; mean-field theory predicts that the curves should coincide. (The overall figure would be perfectly periodic in $\tilde{\alpha}_s$ if the individual curves were each plotted over the full range of the figure.)

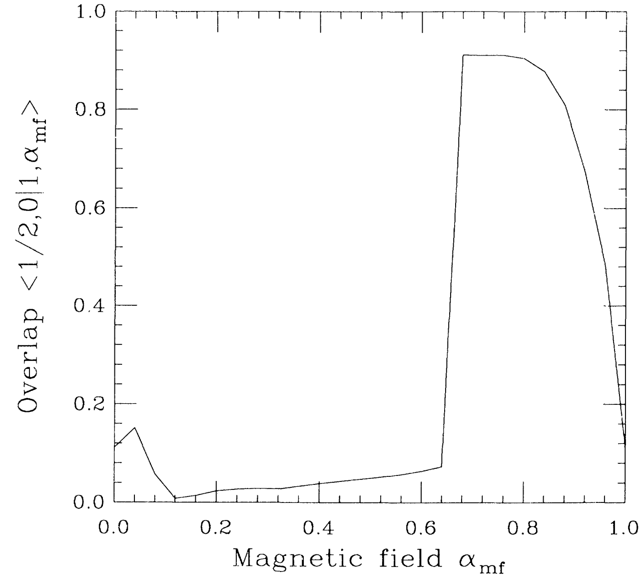


FIG. 2. Overlap $|\langle \frac{1}{2},0|1,\alpha_{mf}\rangle|$ of semion GS with that of fermions in a magnetic field. Mean-field theory (MFT) of statistics predicts a good overlap at $\alpha_{mf}=0.75$.

parameters. Then a MFT of anyons predicts that $|\langle \alpha_s,0|1,\alpha_{mf}\rangle|$ (or $|\langle \alpha_s,0|0,\alpha_{mf}\rangle|$ in the Bose representation) should be large just where $\tilde{\alpha}_s$ of the MFT ket is equal to $\alpha_s$ of the anyon bra. As a test case, consider, for the 3×4/6 system, $|\langle \frac{1}{2},0|1,\alpha_{mf}\rangle| \equiv M$ (Fig. 2). According to MFT, $M$ should be large when $\alpha_{mf}=-\frac{1}{4}, \frac{3}{4},\ldots$. We find in fact that, as we vary $\alpha_{mf}$ between 0 and 1, $M(\alpha_{mf})$ is sharply peaked around $\alpha_{mf}=\frac{3}{4}$, with $M(\frac{3}{4})=0.904$. If we choose to represent the semions $|\frac{1}{2},0\rangle$ via the Bose MFT $|0,\alpha_{mf}\rangle$, we again find that the resulting overlap $M'$ is strongly peaked,[14] with $M'(\frac{1}{4})=0.869$.

The good match of the fermion MFT to real semions, as shown in Fig. 2, can be generalized to various fractional statistics, in both the Bose and Fermi representations. In Fig. 3 we plot $M=|\langle \alpha_s,0|0,\alpha_{mf}\rangle|$ as a function of $\alpha_s$ and $\alpha_{mf}$, for the 4×4/8 lattice. Figure 3 is a contour plot, showing high peaks in $M$ surrounded by steep "cliffs" (the dark regions of closely spaced contour lines). From MFT we expect a ridge of good overlap where $\tilde{\alpha}_s$ as for the vertical axis equals $\alpha_s$ for the horizontal axis; and, based on Eq. (1), the ridge should extend from the origin along a line of slope $n_a$. These expectations are in fact borne out in Fig. 3. One can distinguish five peaks in $M(\alpha_s,\alpha_{mf})$, corresponding to the following wave functions having good overlap: (i) $M(0,0)$, which is trivially 1 at the origin—but note the large size of the "plateau" around this state—only 25% of it is shown; (ii) $M(\frac{1}{2},\frac{1}{4})$, as for 3×4/6; (iii) $M(\frac{2}{3},\frac{1}{3})$, where the Bose representation gives a good MFT of $\frac{2}{3}$ statistics; (iv) $M(\frac{1}{3},\frac{1}{6})$, which is weak; (v) $M(\frac{1}{2},\frac{3}{4})$, a point which is not on the ridge, corresponding to $\tilde{\alpha}_s=\frac{3}{2}$ in the Bose wave function. At these peaks [except at (i), where $M\sim 1$] $M$ takes values in the range 0.5–0.7; in the "troughs" $M<10^{-2}$. We note that the bosons are *not* good at representing fermions. This is

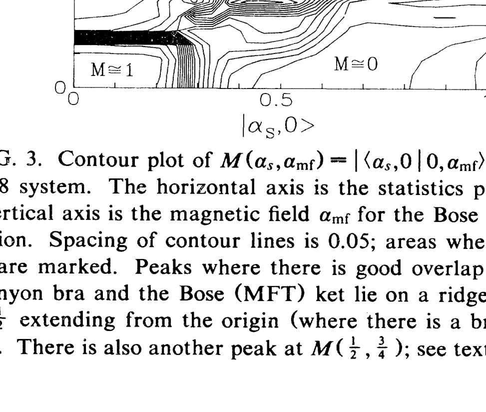


FIG. 3. Contour plot of $M(\alpha_s,\alpha_{mf})=|\langle \alpha_s,0|0,\alpha_{mf}\rangle|$ for the 4×4/8 system. The horizontal axis is the statistics parameter $\alpha_s$; vertical axis is the magnetic field $\alpha_{mf}$ for the Bose GS wave function. Spacing of contour lines is 0.05; areas where $M\sim 0$ or 1 are marked. Peaks where there is good overlap between the anyon bra and the Bose (MFT) ket lie on a ridge of slope $n_a=\frac{1}{2}$ extending from the origin (where there is a broad plateau). There is also another peak at $M(\frac{1}{2},\frac{3}{4})$; see text.

reflected in the small overlap at the right end of the ridge in Fig. 3, and in the structure in $E_{GS}(\tilde{\alpha}_s)$ around $\tilde{\alpha}_s=1$ in Fig. 1. It seems that the $|1,0\rangle$ state is in some sense "repulsive" in parameter space, while the boson states $|0,l\rangle$ (see below) are attractive, as evidenced by the size of the plateau.

We have constructed a similar plot of $M$ for the Fermi MFT of anyons, and the Bose and Fermi plots for the $3\times4/6$, $3\times4/8$, and $4\times4/6$ systems. We find that the Fermi analog of Fig. 3 (i.e., $|\langle\alpha_s,0|1,\alpha_{\rm mf}\rangle|$ ) is very similar, with the ridge appropriately displaced; however, there are fewer separately resolved plateaus, namely, Bose (which is large—see below), semions, perhaps $\frac{3}{4}$ statistics, and of course, fermions. For the smaller lattice the ridge becomes even more obviously a sequence of rather large, rectangular plateaus (like that at the origin in Fig. 3), indicating that the smaller lattice has a more limited power to "resolve" different statistics. We note that the degree of GS overlap with MFT is *not* degraded by deviation from half filling, as is the energy matching.

It is implicit in the notion of a MFT of anyons that magnetic flux renormalizes statistics in two dimensions. On our lattices, HC bosons always have (as expected) a lower GS energy than fermions, for zero magnetic field. From MFT it is clear, however, that fermions in an appropriate magnetic field can have $\tilde{\alpha}_s=2$, and a correspondingly low, Bose-type GS energy (Fig. 1) and wave function. To test this last possibility we have calculated the overlaps between Bose wave functions at $\alpha_{\rm mf}=0$ or 1, and Fermi GS wave functions at various $\alpha_{\rm mf}$, for all four systems described above. The results are unambiguous: The Fermi systems with $\tilde{\alpha}_s=2$ show a large overlap with the boson GS. If we define $\nu$ ($=n_a\lambda/\alpha_{\rm mf}$) as the ratio of particles to flux quanta, then these Fermi systems are characterized by $\nu=1$. In other words, we see the $\nu=1$ quantum Hall effect[5] (QHE) identified in our results as that state in which fermions are renormalized to effective bosons, by the cancellation of phase shifts due to statistics and magnetic field. This idea is consistent with the mapping from Fermi to Bose statistics used by Girvin and MacDonald to characterize the fractional QHE condensate,[15] and with more recent work by Zhang, Hansson, and Kivelson[16] and by Read.[17] We note that we have not calculated transport properties for our $\nu=1$ state; nor have we tested for the size of the excitation gap. We identify this state as the quantum Hall state based on (i) a strong minimum in the energy versus $\alpha_{\rm mf}$ at $\nu=1$, which we expect to become a cusp in the thermodynamic limit; and (ii) the singular-gauge off-diagonal long-range order,[15] which is implied for this state by the good overlap with the bosonic wave function.

By these same criteria, we can also identify the fractional QHE states, for simple fractions ($\nu=1/m$, with $m$ an odd integer), as states in which the fermions become bosons in MFT. These states are, in fact, trivially obtained from the $\nu=1$ state (for $n_a=\frac{1}{2}$) by simply adding an integer number of flux quanta to each plaquette; thus, our lattice model is too coarse grained to capture the important differences between $\nu=1/m$ and $\nu=1$. This disadvantage becomes an advantage if we wish to simulate hierarchical[4,18] QHE states, since we can collapse all the degrees of freedom of the background fluid into a uniform "magnetic field" seen by the quasiparticles (which, being $2\pi$ vortices, "see" electrons as flux quanta[3,18]), while treating the quasiparticles (qp) in the fractional statistics representation.[3,4] We then expect hierarchy states to occur when $\tilde{\alpha}_s$ of the qp, as renormalized [Eq. (1)] by the background "field," is bosonic, or $2n$ (i.e., $p$ in Haldane's notation, or $2p_{s+1}$ in Halperin's). As an example, we have found a deep minimum in the GS energy of $|\frac{1}{3},\alpha_{\rm mf}\rangle$ states, for the $4\times4/8$ lattice, at $\alpha_{\rm mf}=\frac{5}{6}$. This corresponds to a qp filling of $\nu_{\rm qp}=\frac{3}{5}$, or, in terms of electrons per flux quantum, the $\nu=\frac{2}{5}$ daughter state. We also find that $|\langle\frac{1}{3},\frac{5}{6}|0,1\rangle|\sim0.95$. We have obtained essentially identical results for the case $|-\frac{1}{3},\frac{7}{6}\rangle$, which corresponds to the quasihole daughter state $\nu=\frac{2}{7}$. Thus we see that, for hierarchy states, there is a commensuration of real flux to electrons in the parent fluid, and of vortices to background electrons, such that each species' statistics is effectively bosonic.

We comment on the effect of the lattice. We expect the GS to be least frustrated for parameters such that

$$\tilde{\alpha}_s=2n\,, \tag{2}$$

with $n$ an integer. For fermions, or for fractional-statistics quasiparticles, this expression defines a commensuration which describes the quantum Hall effect. In the present of a lattice, however, adding an integer number of flux quanta to each plaquette gives an identical GS energy, and a GS that is the same to within a gauge transformation. Thus, $|0,0\rangle$ is equivalent to $|0,l\rangle$ for integer $l$; however, the latter only satisfies Eq. (2) at half filling. Away from half filling, the two constraints— Eq. (2) and

$$\tilde{\alpha}_{\rm mf}=n_a\tilde{\alpha}_s=l\,, \tag{3}$$

cannot be satisfied simultaneously. In other words, unlike the continuum case, there are two possible commensurations which may determine the GS: net flux to particles [Eq. (2)] or net flux to plaquetes [Eq. (3)]. Thus the unlimited range (in $\tilde{\alpha}_s$) of good match in Fig. 1 is naturally more limited for cases in which $n_a\neq\frac{1}{2}$, since then the periodicities of $\alpha_s$ (period of 2) and of $\alpha_{\rm mf}/n_a$ (period of $1/n_a$) do not coincide, and the energy match is degraded. In the continuum case, only one type of commensuration—flux to particles, Eq. (2)— is meaningful. Hence we view the half-filled case, where the flux—plaquette commensuration does not conflict with that of flux to particles, as being most nearly representative of the continuum case.

Finally, we comment on finite-size effects. The true ground state of the anyon gas is expected to be gapless[9,11] in the thermodynamic limit. Such a state, which is subject to long-wavelength density fluctuations, is not as easily modeled by MFT as is an incompressible state. Our results can be taken to imply that MFT can model short-range correlations fairly well. The problem of long-wavelength fluctuations is not well addressed by finite-size calculations. We note, however, that the existence of a gapless mode has been derived from perturbation theory, starting with the mean-field state.[9,11] This result, and the present results, give some evidence that the MFT of statistics can be an appropriate starting point for theories of anyons.

We are grateful for helpful discussions with Catherine Kallin, Weikang Wu, and D. Yoshioka. This work was partially supported by NSF Grant No. DMR-8802383; computations were performed on the Cray X-MP/48 at the National Center for Supercomputing Applications at the University of Illinois at Urbana–Champaign.

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