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Chaotic but Regular Posi-Nega Switch among Coded Attractors by Cluster-Size Variation

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A globally coupled map lattice is investigated. A simple coding of many attractors with clustering is shown. Through the coding, the attractors are organized so that their change exhibits the period-doubling bifurcation. By a simple input on a site, we can switch among attractors and tune the strength of chaos. A threshold on the cluster size is found beyond which a peculiar "posi-nega" switch occurs.

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The study of coupled chaotic systems is important not only as a model for nonlinear spatially extended systems but also from the viewpoint of biological information processing and possible engineering applications. Coupled map lattices (CML) have been proposed as simple models for spatiotemporal chaos and been extensively investigated.[1-6]

Here we investigate the following globally coupled map lattice:[7]

$$x_{n+1}(i)=(1-\epsilon)f[x_n(i)]+\frac{\epsilon}{N}\sum_{j=1}^{N}f[x_n(j)]\,, \qquad (1)$$

where n is a discrete time step and i is the index of an element ($i=1,2,\ldots,N=$ system size). This is an extreme limit of long-range coupling and a mean-field-theory-type extension of previous CML's.[1-6] We choose the logistic map $f(x)=1-ax^2$, as a prototype for a system of globally coupled chaotic systems.

The questions addressed to this system are:

(A) Does this model give a kind of "mean-field theory" for the rich variety of phases in the pattern dynamics in CML (Ref. 6) and give any better understanding of the transition?

(B) If the frozen random state in CML is related with a "glassy" phase,[6] can the model (1) play a similar role for the state as the "Sherrington-Kirkpatrick model" has played for spin glasses (SG)?[8] Is there any significant difference between our frozen state and SG?

(C) Is there a way to organize many attractors (some of which are chaotic and others of which are periodic)? Are there any bifurcationlike phenomena associated with the change of attractors?

(D) Is it possible to "code" the attractors, so that we can switch among attractors as we like through a simple input? If so, is there any interesting behavior with possible application to information processing?

All of these questions are answered in the affirmative in the present Letter and subsequent papers.[9] Here we study a case where the attractors are well organized and the "switch" among attractors is regular ("posi-nega" switch) even in the presence of chaos.

First, we discuss the possible types of attractors in our model (1). The simplest attractor [type (i)] is the one with $x(i)=x(j)$ for all i,j, in which case the motion is governed just by the single logistic map. The stability of the attractor is easily calculated from the products of N-dimensional Jacobi matrices for the mapping (1), $f'(x_n)[\epsilon/N+(1-\epsilon)\delta_{i,j}]$. From simple algebra, we get the stability condition for the coherent attractor, $\lambda+\ln(1-\epsilon)<0$, where λ is the Lyapunov exponent of the logistic map.

Besides the above single-clustered coherent attractor, we have the following two types of attractors with clustering, that is, the system splits into k clusters, and $x(i)=x(j)$ for $i,j\in$ the same cluster: (ii) Attractors with a small (much smaller than N) number of clusters. (iii) Attractors with a large number of clusters (of the order of or equal to N).[10]

If the elements i,j belong to the same cluster, the dynamics of $x(i)$ and $x(j)$ is governed by the same dynamics, as is seen from Eq. (1). Thus our dynamics (1) can be replaced by the following k-dimensional map after

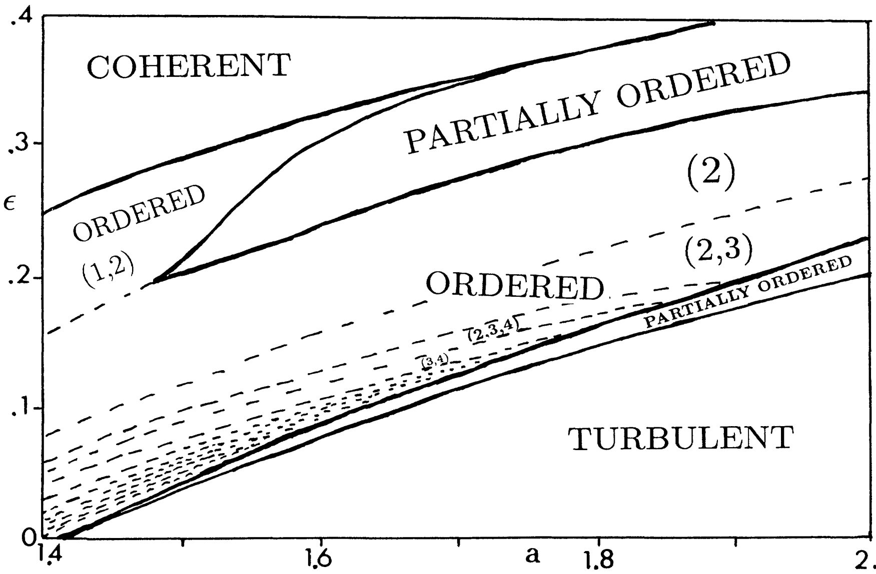

FIG. 1. Phase diagram of our model (1): Phases are determined by the basin volume ratio for k-cluster attractors. The ratio is calculated from the number of initial conditions which lead to a k-cluster attractor. Calculated from 500 randomly chosen initial conditions. $N=200$. The parameter a is changed from 1.4 to 2 by 0.01, while ϵ is changed by 0.02. The numbers such as (1,2,3) show dominant cluster numbers whose basin volume ratio is more than 10%.

our system falls on the k-cluster attractor:

$$X_{n+1}^{\nu}=(1-\epsilon)f(X_n^{\nu})+\sum_{\mu=1}^{k}\epsilon_{\mu}f(X_n^{\mu}), \tag{2}$$

where X_n^{ν} denotes the value of x_n in the νth cluster, and the "effective coupling" ϵ_{μ} is given by $\epsilon_{\mu}=\epsilon N_{\mu}/N$. N_{μ} is the number of elements which belong to the μth attractor.

By taking a randomly chosen set of initial conditions, we can calculate the basin volume for each class of attractor.[11] "Phases" are classified from the basin volume for attractors of each type: (1) Coherent phase: Almost all initial conditions lead to the coherent attractor. (2) Ordered phase: Type-(ii) attractors take almost all basin volumes. (3) Partially ordered phase: Both type-(ii) and type-(iii) attractors have basin volumes. (4) Turbulent phase: Type (iii) takes almost all basin volumes. The phase diagram is shown in Fig. 1, where the conditon of "almost all" is judged by the basin volume ratio larger than 90%.[12] We note the large region of ordered phase.

Hereafter we discuss the case with $\epsilon=0.3$ in more detail. After a transition from the coherent phase, we have found all of the three types of attractors, for $1.6 \le a \le 1.72$, although the attractor of type (ii) has a large basin volume (more than 50%). For $a>1.74$, the basin volume for two-cluster attractors occupies a large ratio, which is more than 95% for $a>1.84$.

In the present Letter, we focus on the parameter region in which two-cluster attractors are dominant. Here the motion of attractors is period-two band (chaotic or periodic). A coherent attractor has a very small basin volume even in the periodic-window regime of the logistic map, where the existence of a stable coherent attractor is assured. The two clusters oscillate out of phase with each other, i.e., in one cluster $x(i)$ changes as $+-+-\cdots$, while the other as $-+-+\cdots$, where $+,-$ is distinguished by whether $x(i)>x^*$ or not, with x^* as the unstable fixed point of logistic map $[(1+4a)^{1/2}-1]/2a$ (this distinction is also used in a short-ranged CML[6]). We define the cluster with $x_{2n}(i)>x^*$ to be "+," and the other "$-$." The number of elements belonging to each cluster is written as N_+ and N_-, respectively ($N_+ + N_- = N$).

The above two-cluster attractors exist for $N-N_{\rm thr} < N_+ < N_{\rm thr}$. The threshold $N_{\rm thr}$ is obtained numerically, and depends on the value a and is proportional to N ($N_{\rm thr}=cN$). The coefficient c increases with a ($c=0.56$ for $a=1.85$ and $c=0.63$ for $a=2.0$). Since there are $N!/(N_+!N_-!)$ attractors for each (N_+,N_-) condition, the number of attractors grows exponentially with N.

How does the state of each attractor change with N_+? As in Fig. 2, $x_{2n}(i)$'s clearly exhibit the period-doubling "bifurcation" to chaos as a function of N_+. Although this looks like a bifurcation, the parameter is fixed here and all that we have done is to arrange the attractors in the order of N_+. In other words, we have found a simple way to organize the many attractors, through which the change of attractors can be seen just as in the bifurcation.

This is not so surprising, since, if we confine ourselves to two-cluster solutions, the dynamics are written by the following two-dimensional coupled map[13] as a special

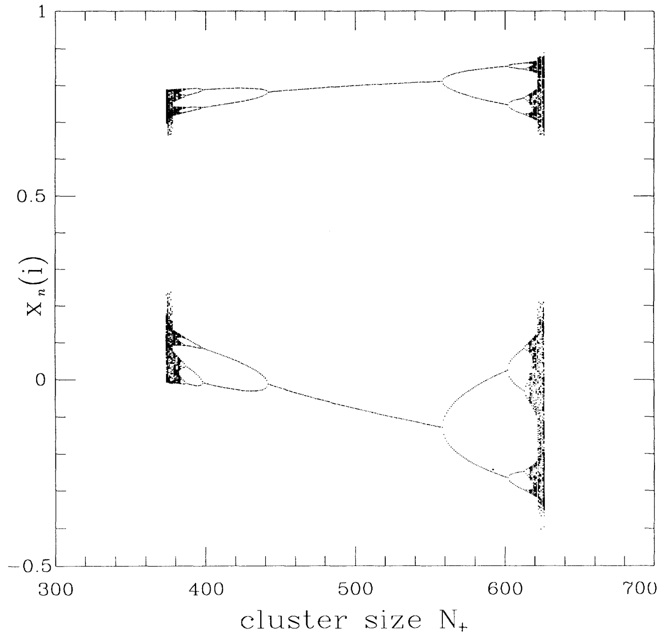

FIG. 2. Change of dynamical state with the change of cluster size: $x_{2n}(i)$'s ($n=2000,2001,\ldots,2260$) are plotted as a function of N_+ for i belonging to the + cluster. $a=1.98$, $\epsilon=0.3$, $N=1000$. $N_{\text{thr}}=626$, as is seen.

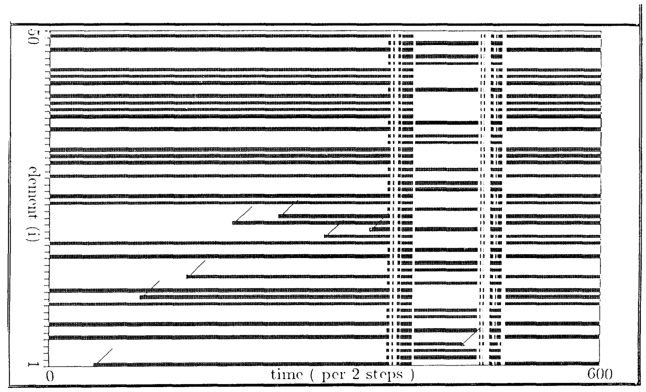

FIG. 3. Site-time diagram: If $x_{2n}(i)>x^*$, the corresponding pixel (i,n) is painted black, otherwise left blank. The arrow indicates the input $\delta=-0.5$ on the corresponding site and time. $a=1.85$, $\epsilon=0.3$, and $N=50$. By successive inputs on site in − cluster, N_+ is increased from 22 to $N_{\text{thr}}=28$, and then a posi-nega switch occurs. Next, by the input to + cluster, the switch again occurs.

case of the k-dimensional map,

$$\begin{aligned} X_{n+1}^{+} &= (1-\epsilon_-)f(X_n^+)+\epsilon_- f(X_n^-)\,, \\ X_{n+1}^{-} &= (1-\epsilon_+)f(X_n^-)+\epsilon_+ f(X_n^+)\,, \end{aligned} \tag{3}$$

with X_n^+,X_n^- as $x_n(i)$ for each cluster, and $\epsilon_\pm=\epsilon N_\pm/N$. Thus the change of N_+ corresponds to the change of bifurcation parameter in the two-dimensional map. We have to note, however, that the above reduction is possible only after the system has fallen onto two clusters. We cannot, for example, obtain N_{thr} by the two-dimensional map.

Next, let us consider the jumping among attractors by an input. We put an input $\delta_n(j)$ onto a single site j, at a single time step n and wait for a while before the system settles down to another (or the same) attractor.[11] If $|\delta|$ is smaller than a threshold, the system goes back to the original attractor after few steps. For larger $|\delta|$, we can make a switch from one attractor to another. By an input on the site j, this site is switched from the + clus-

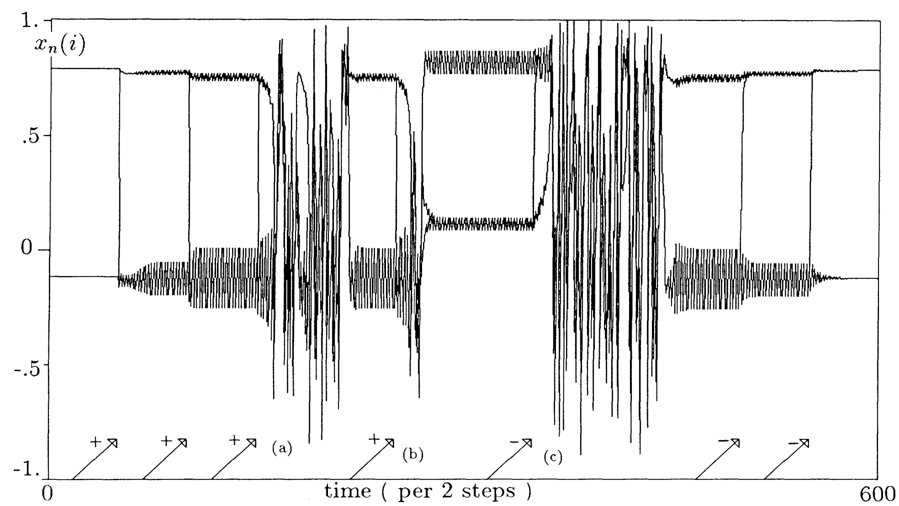

FIG. 4. The time series with switches among attractors. $x_{2n}(i)$'s for all i are plotted as a function of time. If there are only two lines, the system has fallen to two clusters at the time step. The arrow (with +/−) indicates the input $\delta=-0.7$ on a site belonging to the +/− cluster. At (a), the system comes back to the original attractor, while at (b) and (c), it exhibits the posi-nega switch. $a=1.9$, $\epsilon=0.3$, and $N=50$. $N_{\text{thr}}=30$.

ter to $-$, or vice versa. Thus we can change N_+, by successive inputs of $\delta_n(j)$. (See Figs. 3 and 4.) In this manner, we can "control" the attractors through an input.

Then, what happens if we try to increase N_+ beyond $N_{\rm thr}$ or decrease below $N-N_{\rm thr}$? Surprisingly, after intermittent-chaotic[14] transients, all $+$ sites change to $-$ and vice versa (see Fig. 3), unless the system comes back to the original attractor. If our dynamics were confined within two clusters, this might not be so surprising, since the dynamics would be governed by the two-dimensional map. However, this is not the case. In the transient time, the system can split into more than two clusters. The above regular posi-nega switch means that even in the transients, the system still has a strong memory of the previous two-cluster state and the channel to any other states of different $+/-$ distributions is not open or too small to be observed. Schematically this is written as "n"$\rightleftharpoons$"$n+1$"$\rightleftharpoons\cdots\rightleftharpoons$"$N_{\rm thr}$"$\rightleftharpoons$(intermittent transient) $\rightleftharpoons$"$N-N_{\rm thr}$"$\rightleftharpoons$"$N-N_{\rm thr}+1$"$\rightleftharpoons\cdots$, with "n" as a two-cluster attractor with $N_+=n$.

In terms of dynamical systems, the switch belongs to the phenomena called "crisis,"[15] but the crisis here is not confined into a two-dimensional phase space, but connects to a higher dimensional space, and then comes back to the original two-dimensional phase.

For some parameter values (e.g., $a=2.0$, $N=30$), the above intermittent state is not transient but an attractor.[16] The control by the input still works, which is schematically written as "n"$\rightleftharpoons$"$n+1$"$\rightleftharpoons\cdots\rightleftharpoons$"$N_{\rm thr}$" $\rightleftharpoons$"chaotic attractor with spontaneous intermittent switches between $N_+=N_{\rm thr}+1$ and $N_+=N-N_{\rm thr}-1$"$\rightleftharpoons$"$N-N_{\rm thr}$"$\rightleftharpoons$"$N-N_{\rm thr}+1$"$\rightleftharpoons\cdots$.

In this Letter, we have discussed the simplest case of the clustering and switch of attractors. The two-cluster phase in our model (1) corresponds to the mean-field limit of the "pattern selection" in CML, in which few patterns are selected by the suppression of chaos.[6] In our present model again, only the two-cluster attractors are selected, with the strong suppression of chaos.

For smaller nonlinearity and/or coupling, we have seen a variety of cluster numbers ($k>2$). In this phase, the coding of attractors requires both the number of clusters k and the number of elements in each cluster $N_1,N_2,\ldots,N_k$. The dynamical state strongly depends on k and $N_1,N_2,\ldots,N_k$. This state corresponds to the mean-field limit of a "frozen random pattern" in CML.[6] It is again possible to organize the attractors and to code the switch.[9] By a switch between the states with different number of clusters, we can change even the relevant degrees of freedom. In the general situation, the switch can depend not only on the attractor but on the internal state $x(i)$ of the attractor. For example, we have observed a fuzzy posi-nega switch, in which the probability of the transition $+\rightleftharpoons-$ at each site is close to, but not equal to, unity.

Chaotic switches between attractors are discussed in the pioneering paper by Davis,[17] where the adaptive change of a parameter is required. In our case, all the attractors are organized so that the switching is possible only by a simple input.

Lastly, is our observation relevant to biological information processing as has already been argued?[1,18] Neural dynamics is a nonlinear system with global coupling, and our model (1) is one of the simplest among the class of dynamical systems. Thus we may hope that our model may capture some of the essential features of neural dynamics. When we look at some of Escher's figures,[19] for example, we wonder what is the "figure" and what is the "background." If we do not decide which is the figure from a higher level (by attention[18]), our mental state wanders, but once we decide which is which, we can understand the figure easily. This wandering state may be similar to our intermittent switch between $+$ and $-$. Since an experiment on the olfactory bulb[20] illustrates the existence of chaos in the searching process, our observation (especially the posi-nega switch) may be relevant to biological information processing.

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