Effective Monopoles in Noncompact Lattice QED

Simon Hands and Roy Wensley[(a)]
Department of Physics, University of Illinois at Urbana-Champaign, 1110 West Green Street, Urbana, Illinois 61801
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In currently used lattice formulations of quantum electrodynamics, even though gauge fields are introduced as noncompact degrees of freedom, the constraints of gauge invariance imply that fermion fields are sensitive to features best described by compact or periodic variables such as magnetic monopoles. Results from a quenched simulation are presented, showing the existence of a percolation threshold for monopole current networks near the chiral-symmetry-breaking transition already known to occur from previous studies. Implications for the mechanism driving the chiral transition are discussed.

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The possibility of a nontrivial strongly coupled continuum limit in quantum electrodynamics (QED) has received much recent attention. In particular, numerical simulations of both quenched[1] and full[2] versions of the theory on the lattice have measured the chiral-condensate order parameter $\langle\bar\psi\psi\rangle$ in the neighborhood of the continuous chiral-symmetry-breaking transition, and yielded evidence for significant deviations from mean-field behavior. A crucial component in these simulations is the noncompact formulation of the gauge variables. The lattice action is

$$S=S_{\text{gauge}}+S_{\text{Fermi}}(e)\,; \tag{1}$$

$$S_{\text{gauge}}=\tfrac{1}{2}\sum_{n\mu\nu}[\theta_\mu(n)+\theta_\nu(n+\hat\mu)-\theta_\mu(n+\hat\nu)-\theta_\nu(n)]^2$$
$$\equiv\tfrac{1}{2}\sum_{n\mu\nu}\Theta^2_{\mu\nu}(n)\,,$$
$$S_{\text{Fermi}}=\tfrac{1}{2}\sum_{n\mu}\{\bar\psi(n)\gamma_\mu\exp[ie\theta_\mu(n)]\psi(n+\hat\mu)-\text{H.c.}\}$$
$$+m\sum_n\bar\psi(n)\psi(n)\,,$$

where the gauge fields $\theta_\mu(n)$ are oriented real variables in the range $(-\infty,+\infty)$ defined on the lattice links, and the Euclidean fermion fields $\psi(n),\bar\psi(n)$ are anticommuting (in practice the staggered formulation is used). Notice that as in the continuum version, the electron-photon coupling e only appears in S_{Fermi}. Because the gauge fields are noncompact, S_{gauge} is invariant under local gauge transformations defined by the group of real numbers R:

$$\theta_\mu(n)\to\theta_\mu(n)-\Lambda(n)+\Lambda(n+\hat\mu)\,,\quad\Lambda(n)\in R\,. \tag{2}$$

However, S_{Fermi} is invariant under a smaller gauge group, namely, $R/Z\sim\text{U}(1)$:

$$\psi(n)\to\exp[-ie\Lambda(n)]\psi(n);\quad\bar\psi(n)\to\bar\psi(n)\exp[ie\Lambda(n)]\,. \tag{3}$$

In other words, regardless of the form of S_{gauge}, the Fermi fields only perceive *compact* gauge interactions, and would be insensitive as to whether a particular $\Theta_{\mu\nu}$ was equal to 0 or $2\pi/e$, for instance. This is inherent to the way gauge invariance is introduced into the discretized action. In principle, therefore, any fermionic correlation function such as $\langle\bar\psi\psi\rangle$ may well be sensitive to gauge configurations particularly associated with the *periodic* nature of the compact formulation, even though such features play no role in the dynamics controlled by S_{gauge} alone, such as in the quenched model considered here. In this Letter we hope to demonstrate that there are indeed periodic "excitations" or "defects" which are important in driving the chiral transition—namely, magnetic monopoles.

In continuum electrodynamics, a magnetic monopole is defined as the end point of a Dirac string, that is, a long solenoid of infinitesimal radius bearing magnetic flux to the monopole's spatial position, whereupon it spreads out equally in all directions to resemble the flux from a point source. Quantum-mechanical considerations dictate that the flux be quantized; the smallest flux which can be carried by a string is thus $2\pi/e$. In a lattice formulation we can say a plaquette is traversed by a Dirac string if the sum of the angles around the plaquette $e\Theta_{\mu\nu}\geq 2\pi$. Such a plaquette is equivalent to the identity under compact gauge transformations, but carries a nonvanishing contribution to the S_{gauge} defined in (1). This motivates the following definition of magnetic charge on a lattice, first used in compact electrodynamics by DeGrand and Toussaint.[3] For every plaquette on the lattice, identify the Dirac string content by

$$e\Theta_{\mu\nu}(n)=e\bar\Theta_{\mu\nu}(n)+2\pi s_{\mu\nu}(n)\,. \tag{4}$$

The integer $s_{\mu\nu}$ determines the strength of the string threading the plaquette, and $e\bar\Theta_{\mu\nu}\in(-\pi,\pi]$. The integer-valued monopole current $m_\mu(\tilde n)$, defined on links of the dual lattice, is then given by

$$m_\mu(\tilde n)=\epsilon_{\mu\nu\kappa\lambda}\Delta_\nu s_{\kappa\lambda}(n)\,, \tag{5}$$

where Δ_ν is a lattice difference operator; i.e., m_μ is the oriented sum of the $s_{\mu\nu}$ around the faces of an elementary cube. This definition implies the following constraint:

$$\Delta_\mu m_\mu(\tilde{n}) = 0 . \qquad (6)$$

This means that the monopole world lines can either form simple closed loops, or wrap around the lattice, i.e., intersect the boundary an odd number of times and close due to periodicity. These loops are spanned by Dirac world sheets, which are defined by the plaquettes dual to those originally enclosing the string. It is important to note that the monopole current defined in this way is crucially dependent on e: The coupling constant is necessary in defining compact degrees of freedom in terms of the noncompact θ_μ.

The influence of monopoles on the phase structure of quenched compact lattice QED has been understood for some time.[3,4] At weak coupling the loops are small and sparse and have little effect on the vacuum, whereas at strong coupling they enlarge and condense to form a spaghettilike plasma which disorders the phase of Wilson-loop correlation functions and renders the vacuum confining. However, there are important differences between the monopoles in the compact theory, and the effective monopoles in the noncompact theory, which are worth noting. In the former case magnetic flux is conserved modulo 2π, so that the Dirac sheet spanning a loop can be shifted about arbitrarily using suitably chosen gauge transformations. Thus the Dirac sheet can cost no action, and we might expect the loop's action to be proportional to its perimeter: This picture is confirmed by a sequence of transformations on the Villain form of the compact action,[4] which yield an interaction term of the form $m_\mu(x) v_{\mu\nu}(x-y) m_\nu(y)$, where $v_{\mu\nu}(r)$ is the (gauge-fixed) Coulomb propagator, which in four dimensions decays as r^{-2}. The loop action is dominated by the chemical potential per unit length $v_{\mu\nu}(0)$, and hence displays a perimeter law. In the noncompact case, the flux defined by $e\Theta_{\mu\nu}$ is absolutely conserved, and the Dirac sheet *does* cost action, although as explained above, S_{Fermi} is completely insensitive to the sheet, feeling only the encircling monopole current. There is no chemical-potential contribution as such, since the action only increases infinitesimally when $e\Theta_{\mu\nu}$ changes from $\pi-\epsilon$ to $\pi+\epsilon$. On this basis we might expect an area law for the effective monopole loops. Unfortunately a transcription to an action written in terms of the loops or even the plaquette variables remains elusive; the $\Theta_{\mu\nu}$ in S_{gauge} are highly correlated due to Bianchi and Gauss-law constraints. Hence we must resort to numerical studies to assess the influence of monopoles in noncompact QED.

We performed numerical simulations using the action S_{gauge} in (1) on a 12^4 lattice with periodic boundary conditions, using the technique described in Ref. 5. We worked in Feynman gauge and generated configurations in momentum space, returning to configuration space via a fast-Fourier-transform routine. Because S_{gauge} is diagonal in momentum space, this method generates a statistically independent configuration at each sweep, and eliminates critical slowing down. For each configuration, the monopole current network for a variety of values of $\beta \equiv 1/e^2$ in the range [0.15,0.31] was determined using the method described above. Numerical studies of the chiral phase transition in the quenched theory[1,5] have shown that this is the range of interest. Once the monopole network was established, we measured the following quantities to try to find what was going on:

(i) The total amount of monopole current $l \equiv \sum_{n\mu} | m_\mu(\tilde{n}) |$. In principle, the current elements $m_\mu(\tilde{n})$ can take on all integer values between -2 and $+2$;[3] however, the overwhelming majority of nonzero links were ± 1.

(ii) The total area of Dirac sheet $a = \sum_{n\mu\nu} | s_{\mu\nu}(n) |$. As explained above, in a compact formulation this quantity has no gauge-invariant meaning, but it does make sense in the noncompact case and might yield some information about the shape of any loops present.

In addition to these relatively simple measurements, we also used a routine devised to trace out individual loops and hence obtain the distribution of loop sizes. Such techniques have proven very successful in studies of compact QED,[6] where it has been shown that in the confining regime the vast majority of the monopole current line length resides in loops which wrap around the lattice many times. This is consistent with the plasma picture described above. Accordingly we also measured the following quantities:

(iii) The total amount of monopole current l_w which forms part of any loop which wraps at least once around the lattice in any direction.

(iv) The total number of loops n_l.

The results over 100 sweeps are shown in Fig. 1. Both $\langle l \rangle$ and $\langle l_w \rangle$ are given as a fraction of the total number of dual links in the lattice N_l. It is clear that monopoles are present in measurable numbers at these values of β; however, the variation of $\langle l \rangle$ with β is smooth and shows little sign of any discontinuity associated with a phase transition. The ratio of sheet area to perimeter $\langle a \rangle \langle l \rangle^{-1}$ varies from just above $\frac{1}{4}$ at weak coupling, which is the expected value if the monopole loops exist as isolated plaquettes, to just below $\frac{1}{2}$ at $\beta = 0.15$, at which value one might expect each plaquette to share on average two edges with others in the sheet, suggesting that loops form in long "streamers." Wrapping loops assert themselves below $\beta = 0.21$, and by $\beta = 0.15$ do indeed dominate the monopole networks. However, the critical region for the chiral condensate lies in the range [0.24,0.30],[1,5] and the only quantity which gives any indication that this is an interesting region is $\langle n_l \rangle$ which peaks smoothly around $\beta = 0.25$.

It would appear that even if monopoles do play an important role in the dynamics of the chiral transition, the criteria we have chosen so far do not reflect this in the same dramatic way that $\langle l \rangle$ (Ref. 3) or $\langle l_w \rangle$ (Ref. 6)

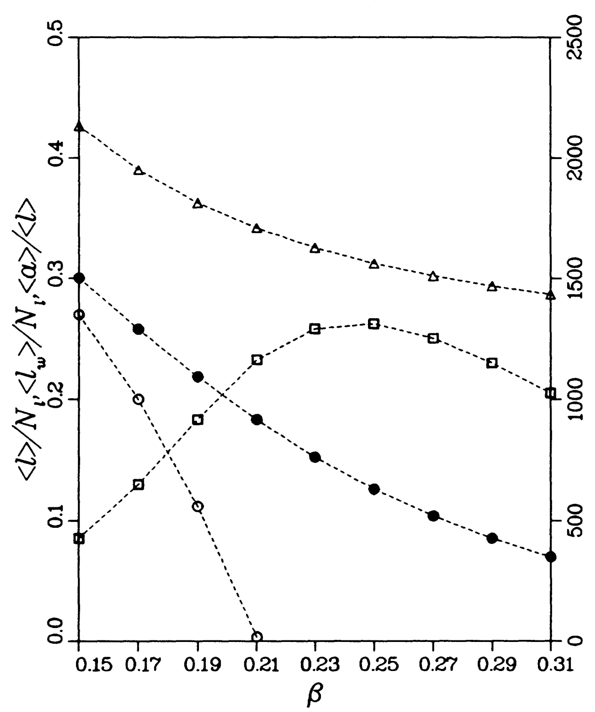

FIG. 1. Plot of $\langle l \rangle/N_l$ (full circles), $\langle l_w \rangle/N_l$ (open circles), $\langle a \rangle/\langle l \rangle$ (triangles), and $\langle n_l \rangle$ (squares) against β.

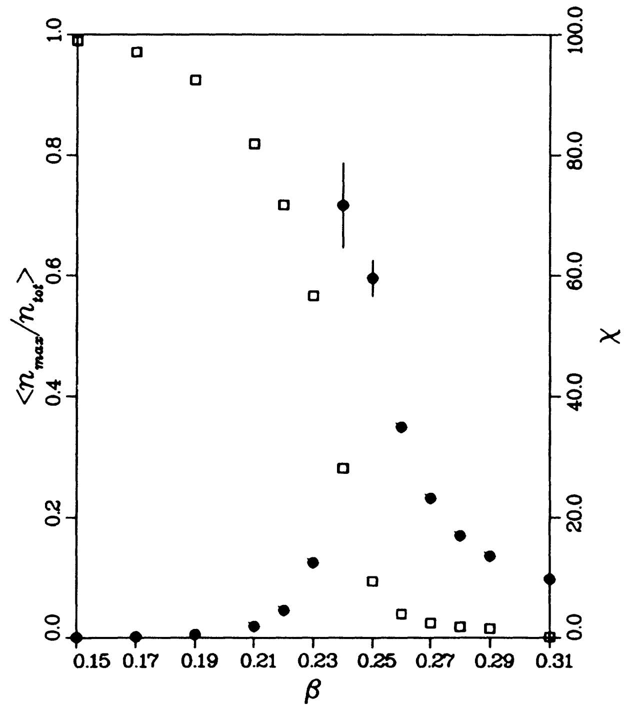

FIG. 2. Plot of probability of connected sites forming part of the largest cluster $\langle n_{\max}/n_{\text{tot}} \rangle$ (squares), and susceptibility χ (full circles) against β.

reflect the confining transition in the compact theory. One explanation could be given in terms of the shapes of the loops in the two theories. If noncompact monopole loops really do obey an area law, then it is plausible that a given loop may intersect itself many times because the action does not favor a large separation of the monopole-antimonopole pair. Under this circumstance the loop tracing routine we used will not necessarily find the longest possible closed path, since at each dual site visited by more than one current line, the program must make a choice about how to continue the loop being traced. We can conceive of a regime where monopole loops are dense enough to extend across the lattice, but the probability of choosing the correct path at each intersection so as to wrap around the boundary is still negligible. This suggests that a more effective measure of monopole activity is to neglect the vector nature of the current altogether and simply count the numbers of dual sites joined into clusters by monopole line elements. It is not difficult to convince oneself that it is then possible to draw a closed monopole loop running through every site of such a connected cluster.

The problem of identifying clusters in this way is known as bond percolation and has a long history in statistical mechanics as an ideal environment for studying critical phenomena. Most studies assume that bonds are occupied randomly with probability p: The basic idea is that at some critical concentration p_c, the largest connected cluster becomes infinite in extent, and begins to occupy a macroscopic fraction of the lattice sites. This point is called the percolation threshold. It is known from series expansions that for four-dimensional bond percolation defined in this way, $p_c \simeq 0.16$.[7] We can see from Fig. 1 that $\langle l \rangle N_l^{-1}$ assumes this value for $\beta \simeq 0.225$, much nearer the range of interest: Moreover, our bonds are not distributed at random but are correlated, both by the (poorly understood) dynamics of S_{gauge}, and by the kinematic constraint (6). This correlation will shift the percolation threshold, if one exists, to even smaller values of $\langle l \rangle N_l^{-1}$.

To test for percolation we ran the system for a further fifty sweeps, this time counting the size n and number of occurrences g_n of all connected clusters at each value of β, using an efficient serial algorithm.[8] In Fig. 2 we display our results for the number of sites in the largest cluster $n_{\max}$ as a fraction of the total number of connected sites n_{tot}, and also the susceptibility χ, defined by

$$\chi = \left\langle \left[\sum_{n=4}^{n_{\max}} g_n n^2 - n_{\max}^2 \right] \Big/ n_{\text{tot}} \right\rangle . \qquad (7)$$

Although neither quantity is uniquely defined, both show unambiguous evidence for a percolation threshold at $\beta \simeq 0.24$, which remarkably is very close to the position of the chiral transition as determined by a mean-field fit to numerical data for $\langle \bar{\psi}\psi \rangle$ at strong coupling[1,5] (the true position of the quenched transition is more difficult to determine, but occurs at a slightly higher value of β). We consider it extremely implausible that these two phenomena are unrelated; there is strong circumstantial evidence that chiral symmetry breaking in lattice QED is driven by the onset of large, tangled loops of effective monopoles. However, we should be wary of looking for a connection with semiclassical formulations,[4] since numerical evidence[1] suggests that chiral symmetry breaking arises from short-ranged effects: Perhaps we should regard the monopoles as lattice artifacts. Even so, it is difficult to reconcile this result with the self-consistent treatments of the continuum gap equation in the quenched planar approximation,[9] which account for the chiral transition without taking account of the compact nature of the fermion-gauge interaction.

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[a]Present address: Department of Mathematical Sciences, St. Mary's College, Moraga, CA 94575.

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