Suppression of the Tearing Mode by Energetic Ions

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The effect of an energetic ion population on the nonlinear stability of a tearing mode of single helicity is determined using kinetic theory. A dynamical equation is derived for the magnetic-island width. It is shown that the island growth can be suppressed in a tokamak when the energetic-ion-density profile peaks just outside the rational surface. Effects due to resistive interchanges and bootstrap currents are included in the model.

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Resistive tearing instabilities are believed to play an important role in disruptions as well as the confinement properties of tokamak plasmas. The first analytic treatment of the nonlinear tearing mode was given by Rutherford,[1] who showed that the island width grows linearly in time when the width of the island exceeds that of the linear tearing layer. The driving force for the instability is the magnetic free energy, which is measured by Δ' in tearing-mode theory.[2]

Previous work has demonstrated that the Rutherford growth, though slower than the linear instability, is rather robust. Kinetic effects[3] or diamagnetic drifts[4-6] do not influence this nonlinear growth. Bootstrap currents provide another source of free energy which enhances island growth.[7,8] Resistive interchange effects also modify the evolution equation of the magnetic island.[9] It has been suggested that by using external helical coils,[10] magnetic perturbations,[11] or radio-frequency currents,[12] it may be possible to suppress island growth.

In this work, we examine the effect of a population of energetic ions on the nonlinear stability of the single-helicity $m \geq 2$ tearing mode. We neglect the magnetic trapping of the energetic ions; this is valid if the hot ions are introduced via parallel neutral-beam injection. We show that the effect of the energetic ions on the nonlinear stability of the mode depends crucially on the local ion-density profile which can be chosen to make the island width much smaller than predicted by a Rutherford analysis.[13] We include effects due to interchange stability and bootstrap currents.

The calculation is done in toroidal geometry; however, to understand the underlying physics of the calculation, we first give a heuristic interpretation of the results in slab geometry. Consider the growth of a magnetic island at the rational surface $x=0$. The sheared equilibrium magnetic field can be written $\mathbf{B}=B_0\hat{\mathbf{z}}+B_y(x/L_s)\hat{\mathbf{y}}$, where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are unit vectors, L_s is the local shear length, and B_0 is a constant magnetic field in the $\hat{\mathbf{z}}$ direction. If a coherent, symmetry-breaking perturbation $\mathbf{B}_1 = b_0 \sin(ky)\hat{\mathbf{x}}$ is imposed, islands of half-width $w=2\times(b_0L_s/kB_y)^{1/2}$ form along the $x=0$ line. This is shown in Fig. 1 which represents the projection of the magnetic field in the $\mathbf{x}$-$\mathbf{y}$ plane.

The guiding-center motion of an energetic ion is given by $\mathbf{v}=v_\parallel\mathbf{b}+\mathbf{v}_d$, where $\mathbf{b}\equiv\mathbf{B}/B$, $B\equiv|\mathbf{B}|$, and $\mathbf{v}_d=[(v_\parallel^2+v_\perp^2/2)/\Omega_i)]\mathbf{b}\times\nabla\ln B$ is the magnetic drift velocity. If we assume that $B=B(x)$ and $B'>0$ (where the prime denotes derivative), the drift velocity is predominantly in the $\mathbf{y}$ direction. If the drift velocity is added to the field-aligned velocity of the equilibrium field, the null line of the velocity component v_y is shifted with respect to the null line of B_y by an amount

$$x_* = -[(v_\parallel^2+v_\perp^2/2)/\Omega_i v_\parallel]B_0L_s/B_yL_B\,, \tag{1}$$

where $L_B=(d\ln B/dx)^{-1}$. Note that the sign of $v_\parallel$ determines whether the null line of v_y is above or below $x=0$. In the presence of the perturbation $\mathbf{B}_1$, the spatial contours of the hot-ion velocity field show islands analo-

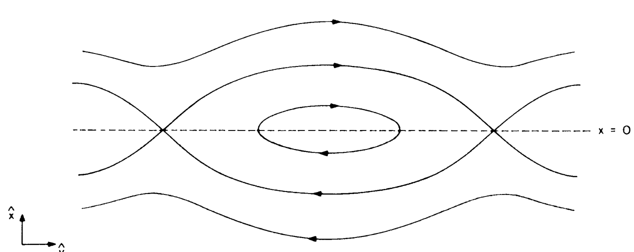

FIG. 1. The projection of the magnetic field in the $\mathbf{x}$-$\mathbf{y}$ plane.

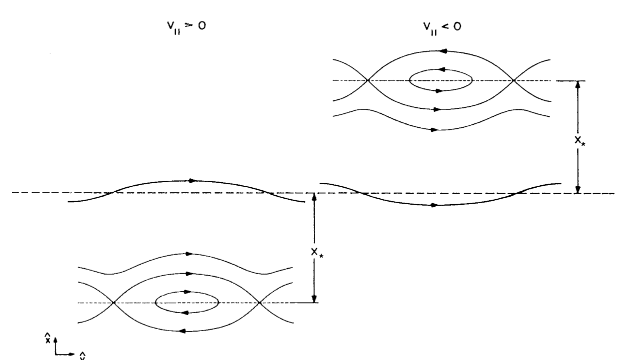

FIG. 2. The projection of the guiding-center velocity field for $v_\parallel=\mathbf{v}\cdot\mathbf{b}>0$ and $v_\parallel<0$.

gous to the magnetic islands of Fig. 1. This is represented in Fig. 2, in which we make the assumption $|x_*| > w$, where w is the island width. The condition $|x_*| > w$ defines the domain of validity for the calculation; we will see that this condition is not difficult to satisfy. Rapid transport of the hot ions along their guiding-center trajectories causes the hot-ion density to be uniform along these trajectories. Hence, the contours in Fig. 2 represent constant-density contours for the hot ions.

If untrapped energetic ions have a net fluid velocity, they produce an electrical current. The plasma electrons tend to follow the ions in order to cancel this current; however, because the electrons scatter into the trapping loss cone at a faster rate than the ions, a net current, which flows in the same direction as the ion flow, is obtained. This is essentially the physical process responsible for current drive using neutral beams.[14,15]

The local influence of the energetic ions is described in Fig. 3. If the inequality $|x_*| > w$ holds, the effect of the perturbing magnetic field on the constant-density contours near $x=0$ is to deform the contours slightly from horizontal lines. We now consider the effect of a hot-ion density gradient. If $n_h' > 0$, there are more ions at the top of Fig. 3 than at the bottom. First, consider the particles with $v_\parallel > 0$. The density gradient causes more ions with $v_\parallel > 0$ to be at the X point of the island than at the O point, because the contour running through the X point lies above the contour running through the O point. This spatial dependence of the ion density introduces a spatial dependence in the net parallel electrical current. This current induces a magnetic field which reduces the perturbing field. The argument for the ions with $v_\parallel < 0$ follows analogously. For $n_h' > 0$, there are more ions with $v_\parallel < 0$ at the O point than at the X point. This spatial dependence also induces a magnetic field that is stabilizing. If the density gradient, or the direction of the magnetic drift (which depends only on B') is reversed, the induced field from the energetic ions enhances the perturbation. Note that this stabilizing effect is independent of the direction of motion of the energetic ions, and depends only on the sign of $n_h' B'$.

We remark that this effect has little influence on the

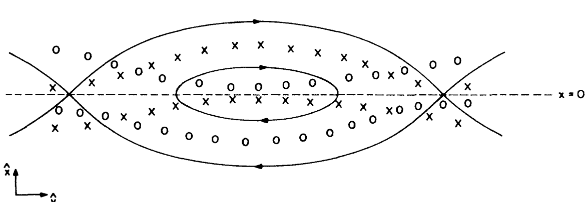

FIG. 3. The interaction of the energetic ions and the magnetic island for $|x_*| > w$. The solid lines are the magnetic field lines, with X's representing constant-density contours for ions with $v_\parallel > 0$ and O's representing constant-density contours for ions with $v_\parallel < 0$ ions.

linear growth of the tearing mode. The extent to which this effect is important depends on the amount of deformation of the constant-density contours, which is proportional to the size of the perturbation. For island sizes smaller than the tearing-layer width, the tearing mode should grow essentially at the rate given by linear theory.

We now provide details of the calculation in toroidal geometry. The magnetic field in toroidal geometry is represented by the flux coordinates Φ, α, and ζ, where Φ is the poloidal-flux function, ζ is the toroidal angle, and $\alpha = \theta - \zeta/q_0$ is the helical angle resonant with the surface $\Phi = \Phi_0$, with $q = q_0$. Near the rational surface, we consider a magnetic perturbation with a single harmonic. The magnetic-flux function in the vicinity of the island is written $\psi = q_0' x^2/2q_0 - A(t)\cos\alpha$, where $q_0' = dq/d\Phi$ evaluated at the rational surface, $x = \Phi - \Phi_0$, and $A(t)$ is a time-dependent amplitude. The half-width w of the island is related to the amplitude A by the equation $w = 2 \times (q_0 A/q_0')^{1/2}$. The magnetic field is written as $\mathbf{B} = q\nabla\Phi \times \nabla\alpha - \nabla\zeta \times \nabla\psi$, where the first term represents the large toroidal field.

An asymptotic analysis of the perturbed Ampere's law in the single-harmonic approximation yields a dynamical equation for the island. As in Rutherford's analysis, the matching condition between the island region and the exterior solution is obtained by integrating the appropriate harmonic across the tearing layer. The matching condition is given by

$$\frac{g^{\Phi\Phi} c q_0'}{16\sqrt{2}\pi R q_0}\Delta' w = \int_{-1}^{\infty} d\chi \oint \frac{d\alpha}{2\pi}\frac{\cos\alpha}{(\chi+\cos\alpha)^{1/2}}\bar{J}_\parallel , \qquad (2)$$

where $\chi \equiv \psi/A$, $g^{\Phi\Phi} \equiv |\nabla\Phi|^2$, and $\bar{h} = \int h\, d\zeta/2\pi$. The parameter Δ' is the discontinuity in the logarithmic derivative of the exterior vector potential. Δ' is obtained from the exterior region, and will be presumed known for the purpose of our analysis.

The analysis for the inner region consists of computing the parallel-current profile which is then used in Eq. (2). The parallel current is obtained from the drift-kinetic equation for each species in the vicinity of the island. As in Ref. 7, we assume that the motion of an electron is collisionless as it circles the major radius, but is collisional around the island. The quantity $\Delta = \nu_e q R/v_t$ is used as a small parameter; here, ν_e is the electron-ion collision frequency, R is the major radius, and v_t is the electron thermal velocity $[v_t = (2T/m_e)^{1/2}]$. The island orbit frequency is $\omega_I \sim (w/a)(qR/v_t)$ and we order $w/a \sim \Delta^2$. The solution for the hot-ion response is obtained from the drift-kinetic equation in two steps. We first impose the ordering $|x_*| \sim w$, and solve the drift-kinetic equation; thereafter, we introduce the subsidiary ordering $w < |x_*|$, as in Fig. 2. The electrostatic part of the electric field is assumed to be of the same order as the electromagnetic contribution, so that $e\tilde{\phi}/T_e \sim \tilde{\phi}/\bar{\phi} \sim \Delta^2$. Finally, we need to order the hot-ion density and energy with respect to the thermal electrons. We take T_e/ϵ_h

$\sim\Delta^2$, $n_h/n\sim\Delta^2$, and $v_h/v_t\sim\Delta$, where ϵ_h, n_h, and v_h represent the energy, density, and velocity of the energetic ions. These orderings enable us to keep track of several important, competing physical effects. However, the physical picture described earlier in this paper is not sensitively dependent on all the specific orderings.

The drift-kinetic equation for the hot ions is now written as

$$(v_\parallel/qR)(q\,\partial_\zeta f_h-[\psi^*,f_h])+(c/q)[\phi,f_h]+ev_\parallel E_\parallel\,\partial_\epsilon f_h=C(f_h)+S\,, \tag{3}$$

where

$$[Q,P]\equiv\partial_\Phi Q\,\partial_\alpha P-\partial_\Phi P\,\partial_\alpha Q\,,$$
$$\psi^*\equiv(q_0'/2q_0)(x-x_*)^2-A(t)\cos\alpha+\rho\,,$$
$$x_*=(v_\parallel+v_\perp^2/2v_\parallel)q_0B_\zeta\partial_\Phi\ln B/q_0'\Omega_i,\quad B_\zeta=\mathbf{B}\cdot\nabla\zeta\,,$$
$$E_\parallel=(1/qR)\{-q\,\partial_\zeta\phi+[\psi,\phi]+(q/c)(\partial A/\partial t)\cos\alpha\}\,,$$

$\partial_\Phi\ln B$ is evaluated at the rational surface and averaged over the torodial angle, ρ is the θ-dependent part of $v_\parallel B_\zeta/\Omega_i(\bar\rho=0)$, and ϕ is the electrostatic potential. The explicit time dependence of the distribution function is ignored since the ions execute many toroidal orbits before the magnetic topology changes. The first two terms account for the field-aligned and the magnetic drift velocities of the energetic ions, whereas the third term represents the $\mathbf{E}\times\mathbf{B}$ drift velocity. The form of the collision operator $C(f_h)$ or the source function S is not very important for what follows, but they can be specialized for the case of neutral-beam injection.[16] The ion collision frequency is ordered according to $\nu_i\sim\Delta^4\nu_e$, so that $\nu_i qR/v_h\sim\Delta^4$.

The hot-ion distribution function is now expanded in a perturbation series in Δ^2, $f_h=\sum_n\Delta^{2n}f_h^n$. To leading order, we find $f_h^0=\bar f_h^0$. To next order in Δ^2, we find $f_h^0=f_h^0(\bar\psi^*)$, where $\bar\psi^*=q_0'(x-x_*)^2/2q_0-A\cos\alpha$. This explains why the contours of Fig. 2 were described as constant-density contours. To determine the velocity dependence of f_h^0, we take Eq. (3) to order Δ^4, and get

$$\langle(qR/v_\parallel)[C(f_h^0)+S]\rangle_*=0\,, \tag{4}$$

where the operator

$$\langle Q\rangle_*=\frac{\int d\alpha\,\bar Q/\partial_\Phi\bar\psi^*}{\int d\alpha(1/\partial_\Phi\bar\psi^*)}$$

defines the drift-surface average.

The quantity of interest is the electrical current generated by the hot-ion distribution near $x=0$. We now impose the subsidiary ordering $|x_*|>w$, so that near $x=0$ we can make the expansion

$$f_h(\bar\psi^*)\cong f_h(x=0)+(q_0A\cos\alpha/q_0'x_*)\partial f_h(x=0)/\partial(x-x_*)\,.$$

The first term essentially gives the exterior density distribution evaluated at the rational surface. We approximate the derivative of f_h as the density-profile gradient at $\Phi=\Phi_0$. The hot-ion distribution then takes the form

$$f_h^0\cong n_h(\Phi_0)V(\mathbf{v})+q_0n_h'(\Phi_0)A\cos\alpha\,V(\mathbf{v})/q_0'x_*\,, \tag{5}$$

where the function $V(\mathbf{v})$ is found from Eq. (4) with the averaging done over constant-Φ surfaces and satisfies the normalization condition $\int d\mathbf{v}\,V(\mathbf{v})=1$.

The electrical current from the hot ions near $x=0$ can now be computed from Eq. (5); we get

$$j_{\parallel h}=\int ev_\parallel f_h\,d\mathbf{v}\cong en_h(\Phi_0)\int v_\parallel V(\mathbf{v})d\mathbf{v}+\frac{en_h'(\Phi_0)\,\Omega_ik_vA\cos\alpha}{B_\zeta d_\Phi\ln B}\,, \tag{6}$$

where $k_v=\int d\mathbf{v}\,V(\mathbf{v})v_\parallel^2/(v_\parallel^2+v_\perp^2/2)$ is a number close to unity. The first term represents the equilibrium parallel current due to the energetic ions. This term will not affect island dynamics. The second term gives the current described earlier in Fig. 3, and has an important effect on the island.

To solve for the electron motion, the drift-kinetic equation is written as

$$(v_\parallel/qR)(q\,\partial_\zeta f-[\psi,f]+[\rho_\parallel B_\zeta,f])+(c/q)[\phi,f]-ev_\parallel E_\parallel\,\partial_\epsilon f=C_0(f)+C_h(f)\,, \tag{7}$$

where $\rho_\parallel=v_\parallel/\Omega_e$. The operator C_0 which represents collisions with the background plasma is taken to be a Lorentz operator, given by $C_0(f)=(\nu\xi/B)\partial_\lambda(\lambda\xi\,\partial_\lambda f)$, where $\lambda=v_\perp^2/Bv^2$, $\xi=\sigma(1-\lambda B)^{1/2}$, $\sigma=\mathrm{sgn}v_\parallel$, and $\nu=\nu_e(v_t/v)^3$. The hot-ion collision operator C_h is written $C_h(f)=-\nu\mathbf{u}_h\cdot\mathbf{v}m_e\,\partial_\epsilon fn_h/2n$, where $\mathbf{u}_h$ is the hot-ion fluid velocity.

The electron distribution is now written as a perturbation series in Δ, $f=\sum_mf_m\Delta^m$. The solution of Eq. (7) to the first four orders gives $f_0=f_{0M}(\psi)$, $f_1=f_{1M}(\psi)$, $f_2=\bar f_{2M}$, and $f_3=\rho_\parallel B_\zeta\partial_\Phi f_0+g_3$, where $g_3=\bar g_3$ and the subscript M denotes a Maxwellian distribution. After taking the toroidal average of Eq. (7) to fourth order in Δ, we obtain

$$g_3=\{(2/qR\nu)(-[\psi,f_2]+eqR\bar E_\parallel f_0/T)+B_\zeta\partial_\Phi f_0/\Omega_e-2u_{\parallel h}f_0/v_t^2\}\sigma vI(\lambda)\,,$$

where $I(\lambda)=\int_{\lambda_c}^{\lambda}d\lambda\,B_0/2|\bar\xi|$ for $\lambda<\lambda_c$, and is zero for $\lambda>\lambda_c$ $[\lambda_c=1/B_0(1+\epsilon)]$, ϵ is the inverse aspect ratio, and B_0 is the magnetic field at the axis. The first two terms represent Ohm's law for the island, the third term represents the contribution to the island bootstrap current, and the last term expresses the electron response to the hot-ion current. Integrating this distribution in velocity space, we obtain the leading-order current due to the electrons,

$$j_{\parallel e}=\sigma_n\left(-\frac{T}{eqR}[\psi,f_2]+\bar E_\parallel\right)-1.46\sqrt{\epsilon}\,\frac{cTB_\zeta}{B}\,\partial_\Phi n-(1-1.46\sqrt{\epsilon})j_{\parallel h}\,, \tag{8}$$

where $\sigma_n=(4.5n_0e^2/\nu_em_e)(1-2.1\sqrt{\epsilon})$ is the conductivity modified to include trapped-electron effects. Trapped-particle effects also produce the bootstrap current and the net current due to energetic ions. The electron density profile, which is needed to compute the island bootstrap current, is determined from the condition that particle flux is constant.[7,9] This gives $\partial_\psi n = q_0 \times n'(\Phi_0)/q_0' w I_1$, where n' is the equilibrium density profile and

$$I_1 = \int d\alpha [(\chi + \cos\alpha)/2]^{1/2}/2\pi .$$

Equations (6) and (8) are added together to yield the total island current. The electrostatic part of the electric field can be associated with resistive-interchange effects. By using the parallel-current constraint, $\nabla_\parallel J_\parallel = -\nabla \cdot \mathbf{J}_\perp$ in a fluid theory, an island current proportional to the resistive-interchange criterion $E+F$ (Ref. 17) can be found.[8] The solution to the parallel-current constraint yields the solution $j_\parallel = j_\parallel(\psi) - \int dl \nabla \cdot \mathbf{J}_\perp$, where l is the coordinate along a field line, with the Ohmic current satisfying the condition $\langle \bar{j}_{\rm Oh} \rangle = (\sigma_n/Rc)(dA/dt)\langle \cos\alpha \rangle$.

The island current is given by

$$j_\parallel = \frac{\sigma_n}{Rc}\frac{dA}{dt}\langle \cos\alpha \rangle + \frac{(E+F)cg^{\Phi\Phi}}{RwI_1}\frac{q_0'}{q_0}(\langle x \rangle - x) - 1.46\sqrt{\epsilon}\frac{cB_\zeta q_0'}{Bq_0}\frac{\partial p}{\partial \psi}\langle x \rangle + 1.46\sqrt{\epsilon}\frac{en_h' A \Omega_i k_v}{B_\zeta d_\Phi \ln B}\langle \cos\alpha \rangle , \tag{9}$$

where

$$\langle P \rangle = \frac{\int d\alpha P/\partial_\Phi \psi}{\int d\alpha (1/\partial_\Phi \psi)} .$$

The first term is the island Spitzer current modified by trapped-particle effects. This term gives the Rutherford result. The island bootstrap current results from the interaction of the banana orbits with the island's pressure profile. The last term is new, and is due to the energetic ions.

Substituting Eq. (9) into Eq. (2), we can obtain a dynamical equation for the magnetic-island half-width. This equation is

$$\frac{1}{\eta_n}\frac{dw}{dt} = k_0 \Delta' - Nw + \frac{Q}{w} , \tag{10}$$

where

$$\eta_n = 1/\sigma_n , \quad k_0 = c^2 g^{\Phi\Phi}/4\pi ,$$

$$N = 1.46\sqrt{\epsilon} k_v \omega_{\rm ph}^2 d_\Phi \ln n_h / 4\pi d_\Phi \ln B ,$$

$$Q = 0.75(E+F)c^2 g^{\Phi\Phi} - 0.5\sqrt{\epsilon} p' R^2 c^2 q_0/q_0' ,$$

with $\omega_{\rm ph}^2 = 4\pi n_h e^2/m_i$. The term Q contains the effects due to resistive-interchange instability and the bootstrap current. The bootstrap current is destabilizing for $p' < 0$, while $E+F$ determines the stability of the interchange mode. The new result of this analysis is the term Nw which enters Eq. (10) and strongly influences the nonlinear stability of the tearing mode. As mentioned, this term is stabilizing if $\mathrm{sgn}(n_h' B') > 0$. For $\Delta' > 0$ discharges, it is possible to control the island width by tailoring the hot-ion density profile. The saturated island half-width is given by

$$w_s = k_0 \Delta'/2N + [(k_0 \Delta'/2N)^2 + Q/N]^{1/2} . \tag{11}$$

Equation (10) is only valid if the inequality $w < |x_*|$ holds. If the island half-width exceeds $|x_*|$, the hot-ion term no longer scales with the island size, and the island grows at the Rutherford rate. The distance $|x_*|$ scales as $\epsilon q_0 v_{\parallel h}/\Omega_i$; for 40-keV protons in a 1-T magnetic field (with $\epsilon \sim \frac{1}{5}$, $q_0 \sim 2$), $|x_*| \cong 0.8$ cm. To guarantee that the island saturates at $w_s < |x_*|$, the energetic-ion parameters must satisfy the inequality

$$v_{\parallel h} \omega_{\rm ph}^2 / L_h \Omega_i > 0.1\sqrt{\epsilon} c^2 |\nabla \Phi| \Delta'/L_s k_v ,$$

where $L_h^{-1} = (n_h'/n_h)|\nabla\Phi|$ and $L_s^{-1} = (q_0'/q_0)|\nabla\Phi|$. For $\Delta' \sim 5/a$, $L_s \sim a$ ($a \cong 20$ cm), this condition is satisfied for $(v_{\parallel h}/\Omega_i L_h) n_h/(10^{12}\ {\rm cm}^{-3}) \gtrsim 1/3$. In particular, the latter condition is satisfied for 40-keV protons in a 1-T field if $n_h \sim 10^{12}\ {\rm cm}^{-3}$ and $L_h \lesssim 6$ cm. Thus, our analysis suggests that it is possible to suppress the $m=2$, $n=1$ island in tokamaks by having the energetic-ion-density profile peak just outside the $q=2$ surface. The energy requirements for the scheme are quite modest; for the parameters given the required neutral-beam energy is approximately 5 kJ which is a small fraction of the energy expended either in Ohmic or neutral-beam heating.

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