Global Strings and Superfluid Vortices

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We explain the relationship between global strings of the Abelian Higgs model and vortices in a superfluid. We show that the nonrelativistic Magnus force law for vortices can be derived from global-string dynamics, but only when an external background field has a special Lorentz-noninvariant configuration $H^{ijk} \propto \epsilon^{ijk}$. We present a self-consistent classical theory for relativistic Higgs vortices in a superfluid, and show that superfluid vortices can be described as a system of spinning global strings.

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A global string is a time-independent vortex solution to the equations of motion of a spontaneously broken global U(1) Higgs model. At low energies there is a single massless degree of freedom in such models, the Nambu-Goldstone boson $\alpha(x)$, defined according to

$$\hat{\phi} \approx \eta e^{i\alpha}, \tag{1}$$

where $\hat{\phi}$ is a complex scalar field and η its vacuum value. As long as we stay away from the core of the vortex, of thickness $\delta \sim 1/\eta$, the Goldstone-boson classical dynamics is governed by two conditions: that it obey the massless wave equation and that it satisfy a constraint that it vary from 0 to $2\pi n$ around any closed contour threaded by n simple vortices. A straight global string at rest is assumed to be a time-independent solution to the equations of motion with α proportional to the azimuthal angle. In flat space the energy density of a global string is $\sim \eta^2/r^2$ close to the core and goes rapidly to zero away. In the vicinity of the string core, Eq. (1) breaks down, $\hat{\phi} \to 0$, and the energy density is $\sim \eta^4$.

It is generally accepted that global strings and nonrelativistic superfluid vortices are closely related; however, the behavior of the two are quite different. For example, a large closed ring of global string collapses and radiates (Fig. 1); a ring of a fluid vortex does not. The discussion here is intended to show that a global string behaves like a vortex in a superfluid only in the presence of a special background field.

Our work fills a gap in a body of theory which has progressed along two directions. One direction was based on thin, structureless strings. The ground-breaking work was that of Lund and Regge,[1] who inferred the Kalb-Ramond string action[2] from the nonrelativistic force law for a superfluid vortex. In this derivation no connection is made with spontaneous symmetry breaking or the Abelian Higgs model. Later Witten[3] and Vilenkin and Vachaspati[4] showed that the static field of a Kalb-Ramond string is on-shell equivalent to the Goldstone-boson field of a global string, satisfying the topological condition for a particular value of the coupling constant. In a previous Letter we showed that the Kalb-Ramond effective action can be derived from the spontaneously broken Abelian Higgs model with a canonical transformation.[5] Here we close the circle by deriving the superfluid vortex force law from the Higgs model.

We begin in the next section by explaining why global strings are inadequate for modeling vortices in a superfluid. In section (2) we derive the relativistic force law for a global string interacting with a background Goldstone-boson field, analogous to the Lorentz force law in

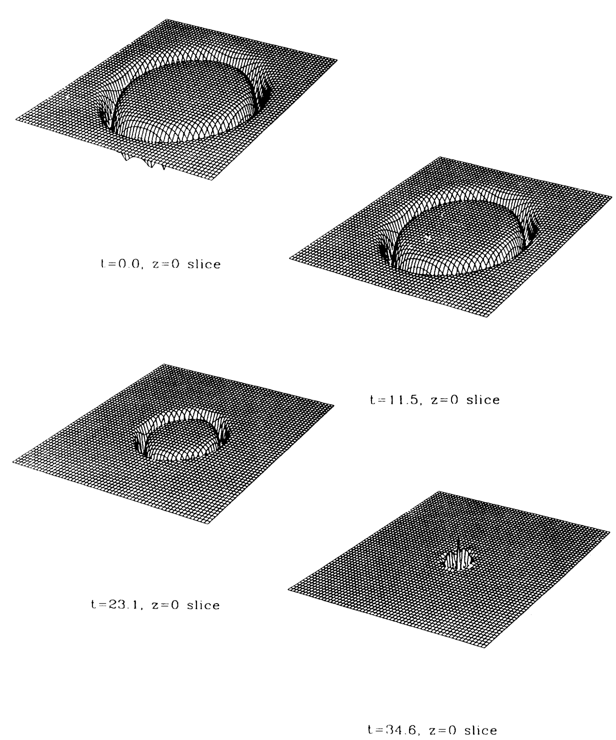

FIG. 1. Computer plots of the collapse of an initially static global string loop. The modulus of the scalar field is plotted for a cross-sectional slice through the loop (Ref. 12).

electrodynamics. In section (3) the solution which corresponds to superfluids is conjectured, and then it is shown that its nonrelativistic limit yields just the observed Magnus force on vortices in a superfluid. In the conclusion we show how the Goldstone-boson field of a superfluid vortex has a linear time dependence, and then discuss some implications for cosmology.

(1) *Nonrelativistic vortices in a superfluid.*—The nonrelativistic equations that govern vortex behavior in a perfect incompressible superfluid at zero temperature are the continuity equation and the Euler equation, and from these one may derive the external Magnus force per unit length[6] on a vortex filament moving with velocity $\mathbf{v}$ through a fluid which, except for the induced velocity field $\mathbf{v}_v$ due to the vortex, is assumed to be at rest:

$$\mathbf{F} = \rho \mathbf{v} \times \kappa . \tag{2}$$

Here κ points along the filament and

$$\kappa = \oint \mathbf{v}_v \cdot dl , \tag{3}$$

for a contour threaded by the vortex. A fundamental property of vortices is that κ is independent of the contour chosen as long as it remains threaded by one string with the same orientation.

It is often said that the vortices in superfluid ^{4}He are similar to the global-string solutions of Eq. (1). Lund and Regge[1] interpreted the dual of the Kalb-Ramond field strength, $\tilde{H}_\mu = \epsilon_{\mu\nu\lambda\rho} H^{\nu\lambda\rho}/6$, as the velocity of a fluid. Since in the Higgs model $\tilde{H}_\mu = \eta \partial_\mu \alpha$, this would mean that the Goldstone boson α is a velocity potential. If this is true then the Magnus force per unit length on the core of a global-string vortex might be estimated by using[7]

$$\kappa \sim \eta^{-1} \oint \partial_\mu \alpha \, dx^\mu = 2\pi \eta^{-1} , \tag{4}$$

and for the fluid density

$$\rho \sim \eta^2/\delta^2 \sim \eta^4 .$$

Given the above, if there is a string segment with radius of curvature R initially at rest then the restoring force per unit length tending to straighten the segment out is $f \sim \eta^2/R$. Setting the magnitude of the resulting Magnus force equal to this gives a limiting velocity

$$v \sim 1/2\pi\eta R .$$

It is then concluded that for $\eta R \gg 1$ the global strings are frozen into the vacuum, unable to oscillate freely, just like superfluid vortices. A large closed loop at rest in vacuum would not collapse.[8]

Yet there are several inconsistencies with thinking that global strings are like superfluid vortices. First, because the energy density falls rapidly ($\sim 1/r^2$) to zero, far from the core of a global string what one has is the vacuum. For superfluid vortices, however, the medium in which the vortices move is a fluid with nonzero energy density and a preferred reference frame. Far away from a string the energy density, rather than going to zero, goes to some nonzero value associated with the fluid. Second, for superfluids the Magnus force is derived using the fact that the vortex carries angular momentum.[6] Near a straight superfluid vortex at rest there is a velocity field moving circularly around it, carrying momentum and kinetic energy. But in a similar configuration a global string is a time-independent solution to the equation of motion. The momentum density away from the string core is

$$\eta^2 \partial_0 \alpha \nabla \alpha ,$$

which is zero in the rest frame of a straight global string.

In short, there seems to be a problem with thinking of global cosmic strings as superfluid vortices. To understand what is wrong we must first study the classical theory of relativistic vortices.

(2) *The Lorentz force law for vortices.*—While it is possible to work with the Goldstone boson $\alpha(x)$ it is more convenient to use an equivalent representation in terms of a two-index antisymmetric tensor $B_{\mu\nu}$, defined at distances far from the core of any vortices by $\eta \partial_\mu \alpha = \epsilon_{\mu\nu\lambda\rho} \partial^\nu B^{\lambda\rho}/2$. In Refs. 1 and 3–5 it was shown, using the equations of motion and comparing at distances large compared to the string core, that the interaction of the vortex with the classical Goldstone-boson field is described by an effective Lagrangian[2]

$$\mathcal{L} = \tfrac{1}{6} H_{\mu\nu\lambda}(x) H^{\mu\nu\lambda}(x) + B_{\mu\nu}(x) j^{\mu\nu}(x) , \tag{5}$$

where $H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\lambda B_{\mu\nu} + \partial_\nu B_{\lambda\mu}$ and $j^{\mu\nu}(x)$ is a source term nonzero only on the vortex worldsheet, defined by

$$j^{\mu\nu}(x) = 2\pi\eta \int \delta^4[x - y(\sigma,\tau)] d\sigma^{\mu\nu} . \tag{6}$$

It is possible to derive Eq. (5) rigorously from a global Abelian Higgs model, with a canonical transformation, uncovering as well the short-distance structure at the vortex core.[5] We take this action as our starting point.

We will derive the relativistic force law for the response of a vortex to the local field $H^{\mu\nu\lambda}$, analogous to the Lorentz force law in electrodynamics. The equations of motion for $B^{\mu\nu}$ are

$$\partial_\mu H^{\mu\nu\lambda} = j^{\nu\lambda} \tag{7}$$

and the stress-energy tensor is

$$\Theta^\mu_\nu = H^{\mu\lambda\rho} H_{\nu\lambda\rho} - \tfrac{1}{6} \delta^\mu_\nu H^2 . \tag{8}$$

Θ^μ_ν is conserved by the equations of motion in the absence of sources. More generally $\partial_\mu \Theta^{\mu\nu} = f^\nu$, where f^ν is an external force density. Using (7) and (8) and the Bianchi identity we find the force law

$$F^\lambda = j_{\mu\nu} H^{\mu\nu\lambda} . \tag{9}$$

Now consider a single isolated string, which may be either closed or infinite. The field strength $H^{\mu\nu\lambda}$ may be split into two parts,

$$H^{\mu\nu\lambda}=H^{\mu\nu\lambda}_{\rm self}+H^{\mu\nu\lambda}_{\rm ext}\,,$$

and in the absence of external background fields the only field is that produced by the string itself. It is easily verified that for a straight string at rest with no background

$$f^{\lambda}=j_{\mu\nu}H^{\mu\nu\lambda}_{\rm self}=0\,,$$

as it clearly must be. Lorentz invariance of the vacuum implies there is no force if it is moving transversely either. In contrast, a perfectly straight string moving through a superfluid does feel a force.[6] It can also be verified from (8) that in the absence of a background field there is no circulating flow of momentum around the vortex, and hence **k** in Eq. (2) does not exist. Straight global strings in vacuum do not feel the Magnus force, Eq. (2).

If the string were not straight or isolated then there would be forces acting on any particular segment because of string-string interactions. If again we assume there is no background field, so that the only fields are those due to the string sources, then as before the energy density falls quickly to zero away from any string core. For example, the condition $H^{\mu\nu\lambda}_{\rm ext}=0$ implies that the field of a single closed loop vanishes at large distances and that the total energy of the system is finite. The force on a segment is given by Eq. (9) using for $H^{\mu\nu\lambda}$ the field caused by other string segments—but this cannot be considered a Magnus-type force. Any sensible generalization of the Magnus force must reduce to Eq. (2) in the special case of a straight isolated string, so this string-string interaction with $H^{\mu\nu\lambda}_{\rm ext}=0$ is not a suitable candidate. In the following we will describe a model with $H^{\mu\nu\lambda}_{\rm ext}\neq 0$ that does lead to the Magnus force in the nonrelativistic limit.

(3) *Relativistic superfluids.*—The fact that the superfluid vortex is immersed in a Lorentz-noninvariant fluid suggests that the correct model for superfluid vortices involves choosing a special background field. Consider a background

$$H^{ijk}_{\rm ext}=\sqrt{\rho}\,\epsilon^{ijk}\,. \tag{10}$$

This is rotation invariant but not Lorentz invariant. From Eq. (8) the energy density and pressure in the absence of vortices are

$$\Theta^{00}=\Theta^{ii}=\rho=p\,.$$

This will be our model for a superfluid in its rest frame.

Consider a string at rest pointing in the $\hat{\mathbf{m}}$ direction. Unlike the cosmological vortex in vacuum, in this background the momentum density has a nonzero contribution

$$\Theta^{0i}=\sqrt{\rho}\,H^{0jk}_{\rm self}\,\epsilon^{i}{}_{jk}\,.$$

Using Eq. (7) to obtain

$$H^{0ij}_{\rm self}=\eta\,\frac{\hat{m}^i r^j-\hat{m}^j r^i}{r^2}\,,$$

we find that there is now an induced energy-momentum flux circulating around the vortex,

$$\Theta^{0i}=(p+\rho)\frac{v^i}{1-v^2}=2\eta\sqrt{\rho}\,\frac{(\hat{\mathbf{m}}\times\mathbf{r})^i}{r^2}\,.$$

Integrating this around a contour threaded by the vortex and dividing by the density gives a vorticity

$$\boldsymbol{\kappa}=(4\pi\eta/\sqrt{\rho})\hat{\mathbf{m}}\,. \tag{11}$$

Now suppose the string is moving with respect to the background. A straight string with constant transverse velocity will again have no self-force. However, there is an external force due to the background field interaction,

$$f=(0,\mathbf{f}),\quad f^i=\sqrt{\rho}\,j_{jk}\,\epsilon^{ijk}\,.$$

The fact that $f^0=0$ implies that it is a conservative force similar to that on a charged particle in a magnetic field.

We would like to develop a fully Lorentz-invariant picture, while still treating the vortex filament as an idealized perfectly thin string. The worldsheet area in Eq. (6) can be written

$$d\sigma^{\mu\nu}=[u^{\mu}\hat{m}^{\nu}-u^{\nu}\hat{m}^{\mu}]\,d\tau\,d\sigma\,,$$

where τ is the proper time of a point on the worldsheet, u is its four-velocity with $u^2=1$, and σ is defined so that $\hat{m}^2=-1$. Consistent with Eq. (11) we also define a four-vector $\kappa^{\mu}=(4\pi\eta/\sqrt{\rho})\hat{m}^{\mu}$. From Eq. (9) the force per unit volume at that point becomes

$$f^{\lambda}=4\pi\eta\,u_{\mu}\hat{m}_{\nu}H^{\mu\nu\lambda}\Delta\,,$$

where Δ is a transverse two-dimensional delta function

$$\Delta(x)=\int\delta^4[x-y(\sigma,\tau)]\,d\tau\,d\sigma\,.$$

Assume a string segment of length L is approximately straight. Taking a small cylinder around the segment, integrating over the volume, and dividing by L gives the net force per unit length

$$F^{\lambda}=4\pi\eta\,u_{\mu}\hat{m}_{\nu}H^{\mu\nu\lambda}=\sqrt{\rho}\,u_{\mu}\kappa_{\nu}H^{\mu\nu\lambda}\,.$$

Now assume also that other segments of string are far enough away that the effect of $H_{\rm self}$ is negligible. In the rest frame of the fluid we then find

$$F^0=0,\quad \mathbf{F}=\gamma\rho\mathbf{v}\times\boldsymbol{\kappa}\,.$$

In the nonrelativistic limit this reduces to the Magnus force, Eq. (2).

(4) *Conclusion.*—The reason why global strings do not behave like superfluid vortices is that global strings live in a locally Lorentz-invariant vacuum. As a result

there is no circulating flow of energy-momentum and hence no fluid vorticity. Superfluid vortices are immersed in a background Lorentz-noninvariant fluid with energy density ρ and a fluid flowing around them. The two objects can therefore be expected to behave quite differently.

We have shown, moreover, that the Magnus force law in a superfluid can be derived from global string dynamics in a specially chosen background field representing the superfluid,[9]

$$H^{ijk} = \sqrt{\rho}\,\epsilon^{ijk} . \tag{12}$$

In the language of the Goldstone boson, this is equivalent to putting

$$\alpha = (\sqrt{\rho}/\eta)t \tag{13}$$

in Eq. (1). A straight string along the z axis in this background would then have

$$\alpha = \varphi + (\sqrt{\rho}/\eta)t ,$$

where φ is the azimuthal angle. We are thus led to a very simple resolution of the question we set out to answer: *A superfluid vortex is a spinning global string.*

Superfluid vortices are known to exist in helium and have been studied extensively. One of the basic properties of these real vortices is that the inertia of the core is negligible and so the core responds instantly to motion in the background fluid. The string moves as if frozen into the fluid, much like magnetic flux lines in a plasma. We can achieve this in our model if we spin the global string fast enough.

Global strings are of cosmological interest and understanding them may be important for axion detection.[10,11] Global strings in most early universe models are immersed in a bath of radiation and matter with density very much less than the density of the string core. The component due to Goldstone-boson radiation, besides being of very low density, is an incoherent bath and will not give a background of the type of Eq. (12). In other words, such cosmological global strings would not be spinning. They will oscillate and radiate and a loop initially at rest will collapse (Fig. 1).

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[9]The authors of Ref. 1 arrived at the Kalb-Ramond action (5) by starting with the nonrelativistic motion of a superfluid vortex line. In their derivation a background field like ours was used as one step, but its relation to the background superfluid was apparently not recognized, and they interpreted the scalar field as a velocity potential [see Eq. (27)ff]. Their subsequent study of the motion of a loop in a static $H^{ijk}_{\rm total} \propto \epsilon^{ijk}$ field does not apply to fluids where the field is really $H^{ijk}_{\rm total} - H^{ijk}_{\rm self} \propto \epsilon^{ijk}$. Our work clarifies these matters as well as draws out the precise relationship to the Abelian Higgs model.

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