Stochastic Manifestation of Chaos in a Fokker-Planck Equation

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We study the behavior of the Floquet spectrum for a Fokker-Planck equation describing a nonlinear Brownian rotor driven by an angle-dependent dynamic external force consisting of two traveling sine waves with amplitudes ϵ_1 and ϵ_2. For $\epsilon_2=0$, the Fokker-Planck equation is separable (in the sense that it has two well defined eigennumbers) and the nearest-neighbor spacing distribution appears to be Poisson random for small spacings. For both $\epsilon_1 \neq 0$ and $\epsilon_2 \neq 0$, we observed nonlinear resonance and level repulsion, indicating that the spectrum, at least locally, exhibits universal random-matrix-type behavior and that information about the underlying dynamics of the Brownian particle is lost.

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The transition to chaos in conservative classical systems has been found to manifest itself in quantum physics as a transition in the nearest-neighbor spectral statistics of the corresponding quantum systems.[1-3] Chaos occurs in conservative classical systems when resonance zones overlap and destroy Kolmogorov-Arnold-Moser (KAM) surfaces which are the remnants of global constants of motion.[4] In the corresponding quantum system, the distribution of spacings between energy levels (quasienergy or Floquet levels for driven systems) satisfies a Poisson distribution (indicating a random distribution of levels) when the classical system is quasi-integrable (dominated by KAM surfaces) because it is a mixture of many independent pure sequences resulting from a full set of quantum numbers. The spacing distribution undergoes a transition to a Wigner or Gaussian orthogonal ensemble (GOE) distribution characterized by level repulsion when the classical system becomes chaotic because constants of the motion (and hence good quantum numbers) are destroyed by resonance overlap. The transition in spectral statistics can occur in local regions of Hilbert space and, as a result, qualitative changes in the behavior of the eigenstates in those regions can occur.[5,6]

In this Letter, we show that classical chaos appears to manifest itself in stochastic physics in a manner very similar to quantum physics. We consider a Brownian rotor which consists of a spherical mass m with radius a attached to a massless rigid rod of length L and zero radius immersed in a fluid with shear viscosity η. The motion of the rotor is constrained to lie in a plane. The Langevin equation[7] describing the Brownian motion of this rotor is

$$I\frac{d\Omega}{d\tau}=-\gamma\Omega+\mathcal{T}_{\text{rand}}(\tau)+\mathcal{T}_{\text{ex}}(\theta,\tau)\,, \tag{1}$$

where

$$\mathcal{T}_{\text{ex}}(\theta,\tau)=\mathcal{T}_1\sin(\theta+\omega\tau)+\mathcal{T}_2\sin(\theta-\omega\tau)\,,$$

$\Omega=\dot{\theta}$ and θ are the angular velocity and angle, respectively, of the rotor at time τ, $I=mL^2$ is the moment of inertia, $\mathcal{T}_{\text{rand}}(\tau)$ is the δ-correlated Langevin torque due to the fluid, $\gamma=6\pi a\eta$ is the Strokes friction, and $\mathcal{T}_{\text{ex}}(\theta,\tau)$ is the torque due to externally applied fields. The random torque is δ correlated,

$$\langle \mathcal{T}_{\text{rand}}(\tau)\mathcal{T}_{\text{rand}}(\tau')\rangle=2\gamma k_B T\delta(\tau-\tau')\,,$$

where T is the temperature, and k_B is Boltzmann's constant. If we set $\gamma=0$ and $\mathcal{T}_{\text{rand}}=0$ in Eq. (1), we obtain Newton's equation for one of the classic models used to study the onset of chaos in both classical[4] and quantum systems.[3,6] The two traveling sine waves induce a self-similar set of nonlinear resonances in the phase space and the classical version undergoes a transition of chaos in the regions where the resonances overlap. The Floquet spectrum for the quantum version of this system undergoes a transition to Wigner-type spectral spacing distribution in a local region of the Hilbert when the resonances overlap.[3] Thus the quantum version of this model exhibits the quantum manifestation of chaos.

We shall consider this system in the limit of a very large Stokes friction, so that the angular velocity of the rotor relaxes to equilibrium on a time scale short compared to the time scales associated with the external torque, $\mathcal{T}_{\text{ex}}(\theta,\tau)$. Then inertial effects may be neglected in Eq. (1) and the equation of motion for the reduced distribution $P(\theta,t)$ (in dimensionless units $t=D\tau$, $\omega_0=\omega/D$, $\epsilon_i=\tau/\gamma D$, $D=k_BT/\gamma$) is given by the Smoluchowski equation[8]

$$\frac{\partial P(\theta,t)}{\partial t}=-\frac{\partial}{\partial\theta}[\epsilon_1\sin(\theta+\omega_0 t)+\epsilon_2\sin(\theta-\omega_0 t)P]+\frac{\partial^2 P}{\partial\theta^2}\,. \tag{2}$$

Thus, the behavior of the system is entirely determined in terms of dimensionless parameters ϵ_1, ϵ_2, and ω_0.

We shall use Floquet theory to rewrite Eq. (2) in the form of an eigenvalue problem. The Floquet theory of such systems was introduced in Ref. 9. We shall use a slightly different form here. The probability density $P(\theta,t)$ satisfies the boundary condition $P(\theta,t)=P(\theta+2\pi,t)$. Thus, we can expand $P(\theta,t)$ in the Fourier series $P(\theta,t)=\sum_{n=-\infty}^{\infty}c_n(t)e^{in\theta}$. The equation of motion for the coefficients $c_n(t)$ is given by

$$\frac{\partial c_m}{\partial t} = -Dm^2 c_m - \frac{m}{2}[\epsilon_1(c_{m-1}e^{+i\omega_0 t} - c_{m+1}e^{-i\omega_0 t}) + \epsilon_2(c_{m-1}e^{-i\omega_0 t} - c_{m+1}e^{+i\omega_0 t})] . \tag{3}$$

Let us introduce the notation $\langle m | c(t)\rangle = c_m(t)$. Then, because Eq. (3) has time periodic coefficients, it will have the following Floquet-type solutions:

$$\langle m | \psi_i(t)\rangle = \sum_{q=-\infty}^{\infty} \langle m,q | \psi_i\rangle e^{-iq\omega_0 t} e^{\Lambda_i t} , \tag{4}$$

where the index i ranges over all Floquet solutions to Eq. (4) and Λ_i is the ith Floquet eigenvalue of the Fokker-Planck equation. The Floquet coefficients $\langle m,q | \psi_i\rangle$ satisfy the following eigenvalue equation:

$$\Lambda_i \langle m,q | \psi_i\rangle = \sum_{n=-\infty}^{\infty} \sum_{q'=-\infty}^{\infty} \langle m,q | \mathcal{W} | n,q'\rangle \langle n,q' | \psi_i\rangle . \tag{5}$$

The quantity $\mathcal{W}$ is the Floquet transition operator for this stochastic process. It is the stochastic analog of the quasienergy Hamiltonian in driven quantum systems.[10] Its matrix elements are given by

$$\langle m,q | \mathcal{W} | n,q'\rangle = (-m^2 + iq\omega_0)\delta_{m,n} - \tfrac{1}{2} m[\epsilon_1(\delta_{n,m-1}\delta_{q',q+1} - \delta_{n,m+1}\delta_{q',q-1}) + \epsilon_2(\delta_{n,m-1}\delta_{q',q-1} - \delta_{n,m+1}\delta_{q',q+1})] .$$

Thus, $\mathcal{W} | \psi_i\rangle = \Lambda_i | \psi_i\rangle$. The Floquet transition matrix $\langle m,q | \mathcal{W} | n,q'\rangle$ is infinite dimensional, complex, and not self-adjoint under Hermitian conjugation. Thus its eigenvalues may be complex and its orthonormal right and left eigenvectors, $| \psi_i\rangle$ and $\langle \psi_i |$, respectively, will not be the same. The vector $| \psi_i\rangle$ is the right eigenvector with eigenvalue Λ_i. We obtain the left eigenvector from the equation $\langle \psi_i | \mathcal{W} = \langle \psi_i | \Lambda_i$. Thus the coefficients $c_n(t) = \langle n | c(t)\rangle$ may be expanded in terms of Floquet states as

$$\langle n | c(t)\rangle = \sum_{i=1}^{\infty} \sum_{q=-\infty}^{\infty} A_i e^{\Lambda_i t} \langle n,q | \psi_i\rangle e^{iq\omega_0 t} , \tag{6}$$

where A_i is determined from the initial conditions on $\langle n | c(0)\rangle$. It is clear from Eq. (6) that in the limit $\epsilon_j \to 0$ $(j=1,2)$, $\Lambda_i \to \Lambda_i^0 = -m^2 + iq\omega_0$. Thus for small coupling the Floquet spectrum is indexed by two "eigennumbers," m and q, which characterize the unperturbed rotor and the degree of excitation of the external field, respectively. For small ϵ_i, the external field, which is harmonic, adds a strong rigidity to the imaginary part of the spectrum. However, as the coupling increases this rigidity relaxes somewhat.

The Floquet matrices for the cases $(\epsilon_1 \neq 0, \epsilon_2 = 0)$ and $(\epsilon_1 \neq 0, \epsilon_2 \neq 0)$ have quite different structure and spectral properties. The Floquet matrix for the case $(\epsilon_1 \neq 0, \epsilon_2 = 0)$ can be reduced to a block diagonal form with an infinite number of infinite-dimensional blocks along its diagonal. Each block connects only those states $| n,q\rangle$ for which $n+q=\alpha$, where α is a constant. [It can be seen from the structure of $\langle m,q | \mathcal{W} | n,q'\rangle$ given below Eq. (5) that the states $| n,q\rangle$ only connect to states $| n+1,q-1\rangle$ and $| n-1,q+1\rangle$.] Thus we can label each block by its value of α given below Eq. (5). When ($\epsilon_1 \neq 0$ and $\epsilon_2 = 0$) the quantity $n+q$ is conserved, whereas when ($\epsilon_1 = 0$ and $\epsilon_2 \neq 0$), $n-q$ is conserved. Thus the spectrum Λ_i for the case $(\epsilon_1 \neq 0, \epsilon_2 = 0)$ is a mixed sequence. It is the superposition of an infinite number of pure sequences, one pure sequence coming from each block. For this particular system, the set of eigenvalues from each block lies along a curved line in the two-dimensional complex eigenvalue plane rather than being scattered throughout a two-dimensional region in the complex eigenvalue plane. Thus we expect to see a Poisson or random nearest-neighbor spacing distribution for this spectrum characteristic of a random process on a line rather than in a plane. The Fokker-Planck equation is separable because even for strong coupling there are two well-defined eigennumbers which label the eigenvalues, Λ_i. One labels the block from which Λ_i comes, and the other labels its position in the sequence of eigenvalues coming from that block.

For the case $(\epsilon_1 \neq 0, \epsilon_2 \neq 0)$, there is no such infinite decomposition of the spectrum. There are two symmetries which allow us to break the Floquet matrix into block diagonal form with four blocks. These four blocks are obtained as follows. Rewrite $P(\theta,t)$ as

$$P(\theta,t) = \sum_{n=0}^{\infty} a_n(t)\cos(n\theta) + \sum_{n=1}^{\infty} b_n(t)\sin(n\theta) .$$

The equations of motion for $a_n(t)$ and $b_n(t)$ completely decouple due to the fact that $\mathcal{T}_{\text{ex}}(\theta,\tau) = -\mathcal{T}_{\text{ex}}(-\theta,\tau)$ yielding a two-block Floquet matrix. Each of these two blocks, which we call sine and cosine blocks, further decomposes into two blocks, one in which the states with $n+q$ odd are coupled and the other in which the states with $n+q$ even are coupled. All information about the long-time behavior is contained in the cosine block.

For the case $(\epsilon_1 = \epsilon_2 = \epsilon \neq 0)$, we have seen evidence of nonlinear resonance and level repulsion. In Fig. 1, we show the values of Λ_i for $\omega_0 = 10$ and three values of ϵ ($\epsilon = 10.0$, 10.2, and 10.5). For these values of ϵ the spectrum still contains a large degree of rigidity. We see that with increasing values of ϵ, the real part of the spectrum (the nonlinear part) appears to resonate and repel. This resonance region runs down the real axis until it reaches the lowest excited state and shifts the real part of larger negative values. (This is the origin of the shift in the real part of the Floquet eigenvalue observed in Ref. 9.) If we increase ϵ still further, the eigenvalues remain shifted upward and become increasingly less rig-

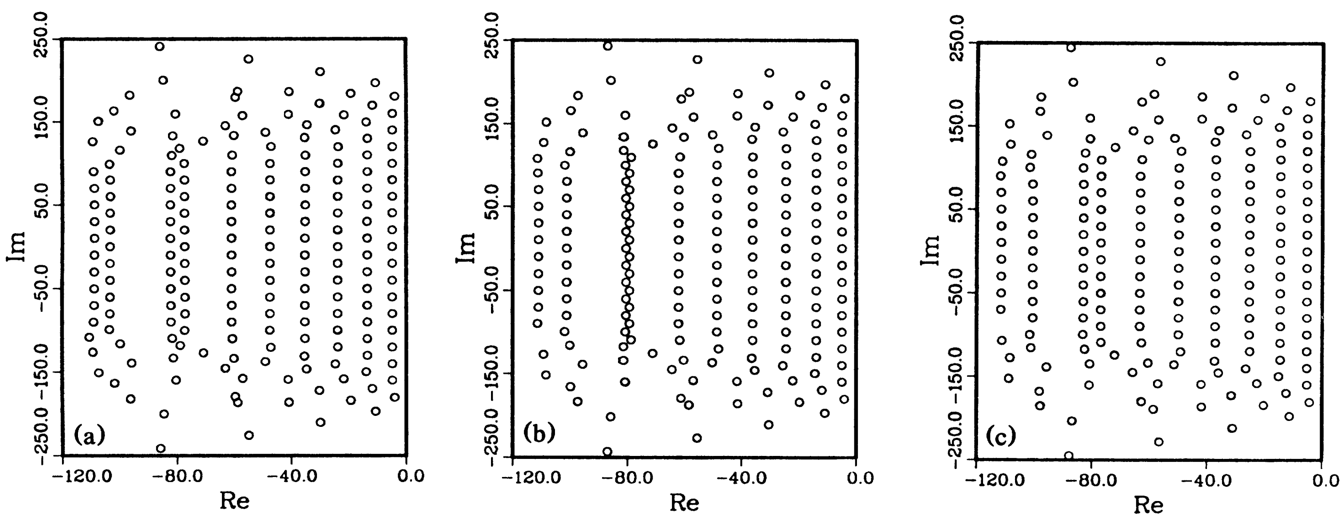

FIG. 1. Distribution of eigenvalues in the complex plane for a 390×390 Floquet matrix. Local level repulsion shows resonance-like behavior. This resonance behavior appears to occur around $\epsilon \approx \omega$ (Ref. 9). We use $\omega = 10.0$ because it is visually more obvious for larger ω. (a) $\epsilon = 10.0$; (b) $\epsilon = 10.2$; (c) $\epsilon = 10.5$.

id. This shifting of the Floquet eigenvalues causes a fairly abrupt drop in the first-passage time for this system. We have constructed the mean-first-passage time by putting absorbing boundaries at $\theta = 0$ and π. We then start the particle at $\theta = \pi/2$. With this choice of boundaries the Floquet spectrum for the first-passage-time problem is determined by the sine block in our Floquet matrix. The average first-passage time is given by

$$\langle T' \rangle = -\left(\frac{2}{\pi}\right)^{1/2} \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \sum_{q=-\infty}^{\infty} B_i \left(\frac{\pi}{2}\right) \frac{\langle n,q \mid \psi_i \rangle}{\Lambda_i + iq\omega_0}, \quad (7)$$

where the coefficient $B_i(\pi/2)$ can be determined from the initial conditions on the Brownian particle. The shifting to more negative values of the Floquet eigenvalues by the resonances should cause a drop in the first-passage time. In Fig. 2, we show the behavior of the first-passage time [obtained numerically using the definition of the first-passage time in terms of the coefficients, $b_n(t)$] as we vary ϵ for $\omega_0 = 1$ and 10. We do indeed see the effects of resonance. The effect of resonance on the first-passage time has also been noted, in a different context in Ref. 7.

We have also studied the spectral spacing statistics for these two cases. For a Poissonian random process on a line, the dimensionless nearest-neighbor spacing s (the nearest-neighbor spacing divided by the average spacing) satisfies a distribution $P_{P1}(s) = e^{-s}$, while for a Poisson random process in a plane it satisfies the distribution $P_{P2}(s) = (\pi/2) s e^{-\pi s^2/4}$. On the other hand, the spectral spacing distribution for asymmetric random matrices with Gaussian distribution exhibit cubic repulsion[11] and satisfy a distribution of the form $P_{RM}(s) = As^3 e^{-Bs^2}$. For a two-dimensional random matrix $A = 3^4\pi^2 2^{-7}$ and $B = 3^2 \pi 2^{-4}$.[11] For high-dimensional random matrices Grobe, Haake, and Sommers[11] have shown numerically that $P_{RM}(s)$ is similar to that for two dimensions but lies slightly below the two-dimensional case for small s and above it for large s. We have computed the spectral spacing statistics of the Floquet eigenvalues for the Fokker-Planck Eq. (2) for the two cases $(\epsilon_1 \neq 0, \epsilon_2 = 0)$ and $(\epsilon_1 \neq 0, \epsilon_2 \neq 0)$ for $\omega_0 = 1.0$. In Fig. 3, we compare the integrated spacing distributions, $I_{Pi}(s) = \int_0^s ds' \times P_{Pi}(s')$ $(i = 1,2)$ and $I_{RM}(s) = \int_0^s ds' P_{RM}(s')$ with the values obtained numerically for the Brownian rotor for $\epsilon \approx 9$. We have diagonalized a 529×529 Floquet matrix ($n \le 23$ and $q \le 23$) for the case $(\epsilon_1 = \epsilon, \epsilon_2 = 0)$. In order to avoid effects due to the finite size of the matrix, only the lowest 130 eigenvalues are used in our analysis. For the case $(\epsilon_1 = \epsilon_2 = \epsilon)$, we have diagonalized a submatrix of size 390×390 ($n \le 20$ and $q \le 20$) and for the same reason only the lowest 110 eigenvalues are used to analyze the spectral statistics. We unfolded the spectrum by multiplying the level spacings, obtained from

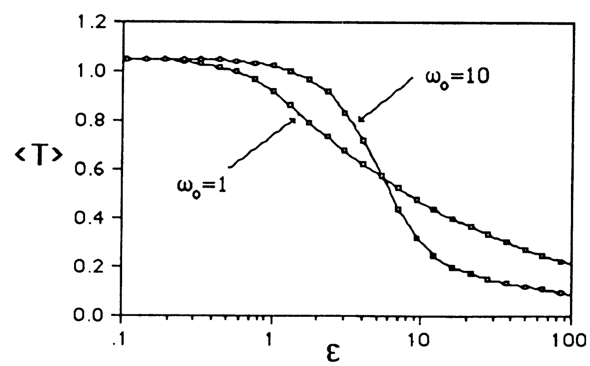

FIG. 2. Plot of mean-first-passage time vs ϵ for $\epsilon_1 = \epsilon_2 = \epsilon$. Here $\langle T \rangle = (2/\pi)^{1/2} \langle T' \rangle$, where $\langle T' \rangle$ is defined in Eq. (7).

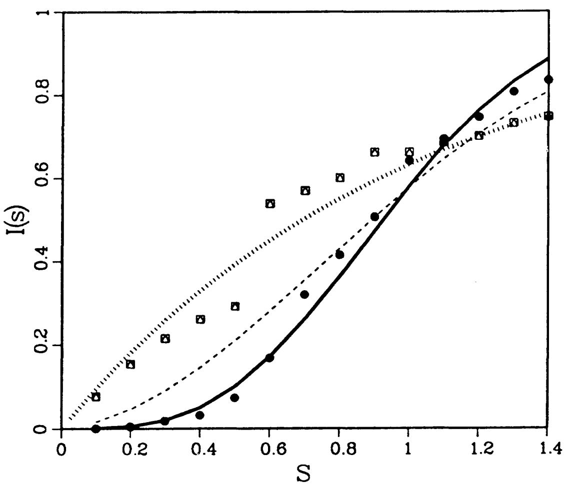

FIG. 3. Nearest-neighbor spectral spacing distribution for $\omega_0 = 1.0$ for the case ($\epsilon_1 = 9$, $\epsilon_2 = 0$) (open squares) and for the case ($\epsilon_1 = \epsilon_2 = \epsilon$) (filled circles). The filled circles represent the average of the four cases $\epsilon = 8.5$ 8.75, 9.0, and 9.25. Also plotted are the integrated spacing distributions, $I_{P1}(s)$ (hatched line), $I_{P2}(s)$ (dashed line), and $I_{\mathrm{RM}}(s)$ (solid line).

the eigenvalue spectrum of the Floquet matrix, by the local average eigenvalue density. With this size matrix we only expect to see strong distortions due to the finite size of the matrix for $\epsilon > 20$. From Fig. 3 it can be seen that the spectral distribution obeys different statistics for these two systems. For the case of single resonance, the spectral spacing statistics follows closely (except for distortions due to rigidity imposed by the harmonic part of the spectrum) a Poisson random process for eigenvalues lying on a line. For the two resonance cases, on the other hand, the spectral statistics approaches cubic repulsion for small spacings indicating loss of one of the eigennumbers and information about the underlying dynamics governing the stochastic process.

In summary, we have studied the Floquet spectral statistics for the Fokker-Planck equation describing a nonlinear Brownian rotor driven by an angle-dependent dynamic external force consisting of two traveling sine waves with amplitudes ϵ_1 and ϵ_2. For the case $\epsilon_2 = 0$, we find that the Fokker-Planck equation is separable and the small spacings appear to be Poisson randomly distributed. For $\epsilon_1 \neq 0$ and $\epsilon_2 \neq 0$, we see evidence of nonlinear resonance and level repulsion of the small spacings, indicating nonnonseparable behavior and loss from the spectrum of information about the dynamics of the Brownian motion process.

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