# Probability Distributions in High-Rayleigh-Number Bénard Convection

Victor Yakhot
*Applied and Computational Mathematics, Princeton University, Princeton, New Jersey 08544*
(Received 5 May 1989)

A theory of probability distributions of temperature in high-Rayleigh-number turbulent Bénard convection is presented.

PACS numbers: 47.25.−c

Recent experiments on high-Rayleigh-number (Ra) Bénard convection conducted at the University of Chicago[1-3] uncovered a new, previously unknown transition between two qualitatively different states of turbulence: At $\mathrm{Ra} \leq 4\times 10^7$ ("soft turbulence") the dependence of the Nusselt number (Nu) on Ra was found to obey the classical $\frac{1}{3}$ law, $\mathrm{Nu} \propto \mathrm{Ra}^{1/3}$. The Nusselt number is defined as the ratio of the heat flux in turbulent flow ($H$) to the heat flux in the conductive regime: $\mathrm{Nu} = HL/\kappa\Delta$, where $\kappa$ is the heat diffusivity, $L$ is the distance between the top and bottom plates, and $\Delta = T_2 - T_1$ is the temperature difference between the plates. At $\mathrm{Ra} \gtrsim 4\times 10^7$ ("hard turbulence") another dependence was observed: $\mathrm{Nu} \propto \mathrm{Ra}^{\beta}$, with $\beta \simeq 0.282 \pm 0.006$. This effect was understood when it was realized that in the hard-turbulence regime the "wind" originating from the energetic large-scale coherent vortex influences the hydrodynamic stability of the thermal boundary layer, thus modifying the mechanism of turbulence production.[1,2] In subsequent experiments[4] on convection in a water tank the role of the wind has been further clarified: It has been shown that turbulence at $\mathrm{Ra} \gtrsim 10^8$ is mainly produced by the plumes emitted by the unstable boundary layer into the bulk of the flow. These plumes can be visualized as relatively long flexible structures with a diameter $\delta \simeq \delta_\beta$ ($\delta_\beta$ is the width of the thermal boundary layer) and with length $L \simeq L_i$ ($L_i$ is the integral scale of turbulence). The typical velocity of a plume is $v_3 \simeq v_w$, where $v_w$ is the velocity of the wind. The existence and role of plumes in turbulence production in Bénard convection was confirmed recently in numerical experiments[5] where transition from soft to hard turbulence has been obtained.

One of the most interesting features of this transition is a dramatic change in the probability density $P(X)$ of the normalized temperature fluctuations,

$$X = \hat{T} - \Theta / \langle (\hat{T} - \Theta)^2 \rangle^{1/2} \equiv T / \langle T^2 \rangle^{1/2},$$

where $\Theta(x,y,z)$ is the mean value of the temperature $\hat{T}$ which is a function of position in the cell. In the soft-turbulence regime ($\mathrm{Ra} \leq 10^7$) the probability distribution measured at the center of the cell is Gaussian, while in the hard-turbulence regime ($\mathrm{Ra} > 10^8$) the observed $P(X)$ was very close to exponential.[3] Similar behavior of $P(X)$ was also obtained in numerical experiments.[5] The exponential dependence of the probability distribution $P(X)$ can be viewed as a manifestation of the intermittent nature of the temperature field in the hard-turbulence regime. Indeed, the flat (flatter than Gaussian) probability density $P(X)$ for large $X$ indicates a high probability of occurrence of large-amplitude events (bursts of turbulent activity) occupying a small fraction of space.

In this Letter the theory of probability distributions developed for the problem of a passive scalar diffusing in a random velocity field[6] is modified for the problem of turbulent Bénard convection.

Let us consider a Bénard cell defined on the domain $-\infty < x < \infty$, $-\infty < y < \infty$, and $-\pi/2 \leq z \leq \pi/2$. The bottom ($z = -\pi/2$) plate is heated by the constant heat flux $H = \mathrm{const}$. In this case $\Theta(x,y,z) \equiv \Theta(z)$ and the equation of motion for the temperature fluctuation $T = \hat{T} - \Theta(z)$ is

$$\frac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T = \kappa\nabla^2 T - v_3 \frac{\partial \Theta}{\partial z} + \kappa \nabla^2 \Theta . \quad (1)$$

Multiplying (1) by $T$ and averaging the resulting equation over the volume leads to

$$\kappa \langle (\nabla T)^2 \rangle = -\langle v_3 T \rangle \partial\Theta/\partial z , \quad (2)$$

which is the well known balance relation equating production and dissipation in convective turbulence.

Multiplying (1) by $T^{2n-1}$, a relation for arbitrary moments of $X$ can be derived:

$$-(2n-1)\langle T^{2n-2} (\nabla T)^2 \rangle = \langle T^{2n-2} v_3 T \rangle \partial\Theta/\partial z . \quad (3)$$

Dividing (3) by (2) gives

$$(2n-1)\langle X^{2n-2} y^2 \rangle = \langle X^{2n-2} y_3 \rangle , \quad (4)$$

where $y^2 = (\nabla T)^2 / \langle (\nabla T)^2 \rangle$, $y_3 = v_3 T / \langle v_3 T \rangle$, and $x^2 = T^2 / \langle T^2 \rangle$; it is clear that $\langle X^2 \rangle = \langle y^2 \rangle = \langle y_3 \rangle = 1$. Assuming that volume averaging is equivalent to ensemble averaging we introduce the probability density

$$P(X, y, y_3) = P(X) q(y, y_3 \mid X) , \quad (5)$$

where $q(\alpha, \beta \mid \gamma)$ is the joint conditional probability of the variables $\alpha$ and $\beta$ for a fixed value of the variable $\gamma$.

Using (5) the relation (4) can be rewritten as follows:

$$(2n-1)\int X^{2n-2}q_1^0(X)P(X)dX = \int X^{2n-2}q_d(X)P(X)dX\,, \qquad (6)$$

where

$$q_1^0(X) = \int y^2 q(y,y_3|X)dy\,dy_3$$

and

$$q_d(X) = \int y_3 q(y,y_3|X)dy\,dy_3$$

are the expectation values of $y^2$ and $y_3$ for a given value of $X$, respectively. It is clear that

$$q_d(X) = \frac{\int y_3(r)\delta[X(\mathbf{r})-X]d^dr}{\int \delta[X(\mathbf{r})-X]d^dr} \qquad (7)$$

and

$$q_1^0(X) = \frac{\int y^2(r)\delta[X(\mathbf{r})-X]d^dr}{\int \delta[X(\mathbf{r})-X]d^dr}\,.$$

If $P(X)X^{2n}\to 0$ when $X\to\infty$, the relation (6) can be recast:

$$\int X^{2n-2}[(P(X)q_1^0(X))'X + P(X)q_d(X)]dX = 0\,. \qquad (8)$$

Equation (8) is satisfied for an arbitrary $n$ only when the expression in the square brackets of the integrand in (8) is equal to zero. This gives the differential equation for $P(X)$ with the solution

$$P(X) = \frac{C}{q_1^0(X)}\exp\left[-\int_0^X \frac{q_d(u)du}{q_1^0(u)u}\right]. \qquad (9)$$

The uniqueness of the solution (9) is proved easily for the case $P(X)=P(-X)$, $q_d(X)=q_d(-X)$, and $q_1^0(X)=q_1^0(-X)$ in which we are interested in this work. The properties of (9) can be analyzed readily in the limit $X\to 0$. Relation (9) gives the probability density $P(X)$ defined by two variables only ($q_d$ and $q_1^0$) instead of the infinite set of moment relations (4).

When $X\to 0$, $q_1^0 \simeq 1+kX^2$ since $q_1^0(X)=q_1^0(-X)$ and $q_1^0(X)$ is assumed to be at least twice differentiable at $X=0$. The case $q_1^0=1$ corresponds to the Kolmogorov theory of turbulence based on the assumption of a constant, nonfluctuating dissipation rate $(\nabla T)^2=$const. Thus, the contribution $kX^2$ to $q_1^0(X)$ can also be viewed as a correction to the Kolmogorov (mean-field) value of $q_1^0(X)=1$ due to the dissipation fluctuations. In this case the relation $q_1^0\simeq 1+kX^2$ might be valid for both large and small values of $X$.[7]

Substituting the definition of $y_3$ into (7) gives

$$q_d(X) = \frac{X}{\langle v_3X\rangle}\frac{\int v_3(\mathbf{r})\delta[X(\mathbf{r})-X]d^dr}{\int \delta[X(\mathbf{r})-X]d^dr}\,. \qquad (10)$$

Relations (9) and (10) show us that the probability distribution $P(X)$ is dominated by the conditional expectation value of $v_3$ for a fixed value of $X(\mathbf{r})=X$. To investigate the analytic properties of $q_d(X)$ we notice that $\langle v_3T\rangle = H =$const has a fixed (positive) sign. This means that choosing $T(r)=T>0$ and considering values of $v_3(r)$ at all points of the flow with $T(r)=T>0$ we shall find that the mean value of $v_3(r)$ *taken over the domain where* $T(r)=T>0$ *is equal to* $v_3(T)>0$. This result is the consequence of the symmetry of the problem: The warm portions of fluid are transferred from the bottom plate by the positive velocity fluctuations $v_3>0$. At the same time the negative temperature fluctuations $T<0$ are carried from the top plate by the negative velocity fluctuations $v_3<0$. Thus, *the conditionally averaged velocity* $v_3(T)$ *is an odd function of* $T$ *or of* $X$, $v_3(X) = -v_3(-X)$.

Two cases are to be considered:

(1) $v_3(X)\propto X$ when $X\to 0$. In this case turbulence production does not involve any particular velocity scale. This gives $q_d(X)\propto X^2/\langle v_3X\rangle$ and the probability density $P(X)$ is Gaussian in accordance with (10).

(2) The turbulence production involves a well-defined velocity scale. For example, the hard turbulence is produced by the plumes emitted from the boundary layer with velocity $v\simeq v_w$. In this case positive and negative temperature fluctuations $X>0$ and $X<0$ are carried by the plumes having velocities $+v_w$ and $-v_w$, respectively, and according to (10), $q_d = v_wX/\langle v_3X\rangle(1+kX^2)$ provided turbulence is dominated by the plumes in the limit $\mathrm{Ra}\to\infty$. Combining this with relation (9) and the expression for $q_1^0(X)=1+kX^2$ gives the following for small $kX^2$:

$$P(X)\propto \exp[-(\alpha/\sqrt{k})\mathrm{arctg}\sqrt{k}X] \simeq e^{-\alpha x}\,, \qquad (11)$$

where

$$\alpha = T_{\mathrm{rms}}v_w/\langle v_3T\rangle = T_{\mathrm{rms}}v_w/H \sim 1\,.$$

The value of $\alpha$ can be estimated readily. In Kolmogorov turbulence $L_i/2\pi = 1$, $\langle v^2\rangle = 3C_K\bar{\epsilon}^{2/3}$, and $\langle T^2\rangle = 2\kappa\langle(\nabla T)^2\rangle\bar{\epsilon}^{-1/3}\mathrm{Ba}$, where the dimensionless constants $C_K$ and Ba are equal to $\simeq 1.6$ and $\simeq 0.8$, respectively. It has been shown in Ref. 5 that $v_w^2 \simeq 0.25\langle v^2\rangle$. Substituting these values into the definition of $\alpha$ we derive $\alpha \simeq (0.25\times 3\times 1.6\times 1.6)^{1/2} \simeq 1.4$. This value is to be compared with $\alpha=1.3$ and $\alpha=1.2$ observed in Refs. 4 and 5, respectively.

It has to be emphasized that according to the exact relations (9) and (10) the exponential probability distributions of the normalized temperature always exist when convective turbulence is dominated by the bursts having a well-defined velocity scale $v\sim 1$. Even in the case of "soft" turbulence, characterized by the Gaussian distribution, an artificially introduced instability of the thermal boundary layer will break the symmetry and cause the transition to the exponential distribution function $P(X)$ provided this instability generates the velocity fluctuations with probability density $P(v_3)$ sharply peaked at $|v_3| = v_0 \sim 1$. This prediction can be tested experimentally by carefully designed mechanical perturbations of the thermal boundary layer to enhance its instability.

This work was supported by the Air Force Office of Scientific Research under Contract No. F49620-87-C-0036, the Office of Naval Research under Contract No. N00014-82-C-0451, and the U.S. Defense Advanced Research Projects Agency under Contract No. N00014-86-K-0759.

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