Limiting Probability Distributions of a Passive Scalar in a Random Velocity Field

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Diffusion of a passive scalar in a random velocity field $\mathbf{v}(\mathbf{x},t)$ is considered. An exact, closed form of the limiting $(t\rightarrow\infty)$ probability distribution of the normalized scalar is derived. The predictions of the theory are compared with the results of numerical simulations.

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The equation of a passive scalar $T(\mathbf{x},t)$ diffusing in a random velocity field $\mathbf{v}(\mathbf{x},t)$,

$$\frac{\partial T(\mathbf{x},t)}{\partial t}+\mathbf{v}\cdot\nabla T(\mathbf{x},t)=\kappa_0\nabla^2 T(\mathbf{x},t),\qquad \nabla\cdot\mathbf{v}(\mathbf{x},t)=0\,, \tag{1}$$

is interesting from both a theoretical and an experimental point of view. This equation describes the evolution of a passive contaminant in a random medium and is used for calculations of heat and mass transfer in turbulent flows. Theoretical investigations of (1) have mainly been restricted to consideration of properties of the scale-invariant solutions provided the scaling properties of the correlations of the velocity field are known.[1-6] Recent experiments on scalar diffusion in turbulence[7,8] and on the temperature fluctuations in Rayleigh-Bénard convection[9-11] led to new experimental data on the probability distribution of a scalar (dye concentration or temperature) in turbulence. The major question yet to be addressed can be formulated as follows: Is there a universal probability distribution of a scalar in a random velocity field? If this distribution exists, then what are its properties? The problem formulated by Eq. (1) has recently been attacked by the renormalization-group method[12-14] which allowed determination of the scaling properties of the solutions of (1). The applicability of the renormalization-group approach to this problem is not so obvious because Eq. (1) describes essentially a nonstationary process. Indeed, let us introduce $Q=\langle T^2(\mathbf{x},t)\rangle$. The equation for Q is derived from (1),

$$\partial Q/\partial t=-2\kappa_0\langle(\nabla T)^2\rangle\,, \tag{2}$$

and thus $Q\rightarrow 0$ as time $t\rightarrow\infty$ provided that $\kappa_0>0$.

Numerical experiments on the time evolution of the passive scalar governed by (1) revealed some interesting features: After a period of relaxation the fourth-order moment of the normalized field, $X=T/Q^{1/2}$, reaches a stationary limit $\langle X^4\rangle=\text{const}$. In this work we study the properties of the stationary probability distributions of the properly normalized passive scalar governed by (1). The exact, closed form for the probability distributions will be derived for the case of arbitrary statistics and dynamics of a random velocity field $\mathbf{v}(\mathbf{x},t)$. The derived formula expresses the probability distribution function (PDF) of the normalized passive scalar in terms of the conditional expectation value of the normalized scalar dissipation rate. This feature of the PDF was noted in an important work by Pope[15,16] who derived a time-dependent equation for the probability density of the scalar, but neither analyzed it in a general form nor gave a closed-form solution. Here we are interested in the properties of the limiting $(t\rightarrow\infty)$ PDF's for which a simple formula can be derived and compared with the results of numerical experiments.

The equation of motion for the variables T^{2n} following from (1) is

$$\frac{\partial T^{2n}}{\partial t}+\mathbf{v}\cdot\nabla T^{2n}=\kappa_0\nabla^2 T^{2n}-2n(2n-1)\kappa_0 T^{2n-2}(\nabla T)^2\,. \tag{3}$$

Next, according to the definition of X,

$$\frac{\partial X^{2n}}{\partial t}+v_i\frac{\partial X^{2n}}{\partial x_i}=\kappa_0\nabla^2X^{2n}-2n(2n-1)\kappa_0\frac{X^{2n-2}}{Q}(\nabla T)^2+\frac{2n\kappa_0X^{2n}}{Q}\langle(\nabla T)^2\rangle\,. \tag{4}$$

Averaging (4) over the space and taking into account that

$$\langle\nabla^2X^{2n}\rangle=\langle v_i\,\partial X^{2n}/\partial x_i\rangle=\frac{\partial}{\partial x_i}\langle v_iX^{2n}\rangle=0\,,$$

the exact relation for the moments $\langle X^{2n}\rangle$ is obtained in the stationary state:

$$(2n-1)\langle X^{2n-2}(\nabla T)^2/Q_1\rangle=\langle X^{2n}\rangle\,, \tag{5}$$

where $Q_1=\langle(\nabla T)^2\rangle$. The most interesting property of (5) is that all moments of the variable X are entirely determined by the correlations of X^{2n} with the single variable $(\nabla T)^2/Q_1$. The parameter $2\kappa_0\langle(\nabla T)^2\rangle=N$ is the mean dissipation rate of the scalar field in accordance with (2). If X^{2n} and $(\nabla T)^2$ are statistically independent so that $\langle X^{2n-2}(\nabla T)^2\rangle = \langle X^{2n-2}\rangle\langle(\nabla T)^2\rangle$, then, according to (5), $(2n-1)\langle X^{2n-2}\rangle = \langle X^{2n}\rangle$ and $\langle X^{2n}\rangle = (2n-1)!!$. This means that X is a Gaussian variable with probability density $P(X)$: $P(X) = \pi^{-1/2} \times \exp(-X^2/2)$. The statistical independence assumption leading to this result is too strong. To verify it we have to consider the dynamical equations for $(\nabla T)^2$ and determine the moments involved in (5). The relation (5) does not explicitly include the velocity field $\mathbf{v}(\mathbf{x},t)$, and thus it is valid for an arbitrary field $\mathbf{v}(\mathbf{x},t)$. It does not mean, however, that the properties of the random field $\mathbf{v}(\mathbf{x},t)$ are not reflected in the probability distribution functions since the variable $\mathbf{v}(\mathbf{x},t)$ does enter the recursion relations for the moments of the scalar derivative $\partial T/\partial x_i$.

To further investigate the properties of the stationary distribution $P(X)$ following from (5) let us introduce a normalized scalar dissipation rate

$$y^2 = (\nabla T)^2/\langle(\nabla T)^2\rangle \equiv (\nabla T)^2/Q_1$$

so that $\langle y^2\rangle = \langle X^2\rangle = 1$ and rewrite (5) as

$$(2n-1)\langle X^{2n-2}y^2\rangle = \langle X^{2n}\rangle . \qquad (6)$$

Introducing the joint probability distribution $P(X,y)$, the relation (6) can be written as

$$(2n-1)\int X^{2n-2}y^2P(X,y)dX\,dy = \int X^{2n}P(X,y)dX\,dy . \qquad (7)$$

Let us seek the solution $P(X,y)$ as $P(X,y) = P(X) \times q(y\,|\,X)$, where $q(y\,|\,X)$ is the conditional probability of the variable y for a given value of the second variable X. Substituting this form of $P(X,y)$ into (7) and using the normalization condition $\int q(y\,|\,X)dy = 1$ leads to

$$\int (X^{2n-1})'P(X)dX\int y^2q(y\,|\,X)dy = \int X^{2n}P(X)dX . \qquad (8)$$

Defining the expectation value of y^2 for given X,

$$q_1^0(X) = \int y^2 q(y\,|\,X)dy , \qquad (9)$$

the relation (8) can be rewritten as

$$-\int X^{2n-1}\left[\frac{\partial P(X)}{\partial X}q_1^0(X) + P(X)\frac{\partial q_1^0(X)}{\partial X} + XP(X)\right]dX = 0 . \qquad (10)$$

The differential equation for $P(X)$ is

$$\frac{1}{P}\frac{\partial P}{\partial X} = -\frac{X}{q_1^0(X)} - \frac{\partial \ln q_1^0(X)}{\partial X} \qquad (11)$$

and thus

$$P(X) = C[q_1^0(X)]^{-1}\exp\left[-\int_0^X \frac{u\,du}{q_1^0(u)}\right] . \qquad (12)$$

The proof of the uniqueness of the solution (12) is simple. In the statistically isotropic case $\langle X\rangle = 0$, $P(X) = P(-X)$ and $q_1^0(X) = q_1^0(-X)$. Thus the function inside the square brackets in the integrand of (10) is an odd function of X. Thus, relation (10) is satisfied only when the expression in the square brackets of (10) is identically equal to zero.

Formula (12) gives the probability distribution of the random variable X. To make further progress, the information about $q_1^0(X)$ is needed. When X and y are two statistically independent variables $q_1^0(X) \equiv 1$ and relation (12) is reduced to the Gaussian distribution function.

The major advantage of the relation (12) is that it allows us to use some basic symmetries of the problem combined with assumptions about analytic properties of $q_1^0(X)$ to investigate the functions $P(X)$. For example, in isotropic, homogeneous turbulence $P(X) = P(-X)$ and $q_1^0(X) = q_1^0(-X)$. Next, let us assume that $q_1^0(X)$ is at least twice differentiable at $X=0$. When the scalar dissipation rate is statistically independent of the scalar itself, we have $q_1^0(X) = 1$. To account for the correlation between these variables, let us expand $q_1^0(X)$:

$$q_1^0(X) \simeq 1 + kX^2, \quad k = \tfrac{1}{2}\,\partial^2 q_1^0/\partial X^2\,|_{X=0} , \qquad (13)$$

where we assume $k \sim 1$.

Substituting this relation into the formula (12) leads to

$$P(X) = \frac{C}{(1+kX^2)^{1+1/2k}} , \qquad (14)$$

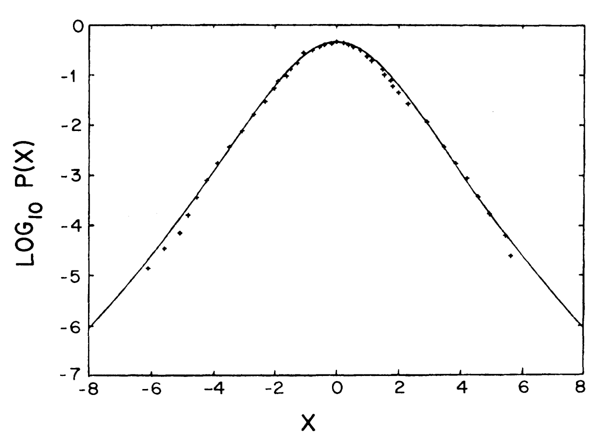

FIG. 1. Logarithm of the probability density of normalized passive scalar as a function of $X = T/T_{\rm rms}$. +, results of numerical simulations; ——, theory based on relation (14) with $k = 0.08$.

where $k \sim 1$. Direct numerical simulations of the problem of a passive scalar were conducted in Ref. 17. The results of the simulations are compared with predictions given by formula (14) in Fig. 1. The agreement between theory and experiment is excellent in the entire range $-7 \lesssim X \lesssim 7$. Understanding why a theory based on an expansion which is valid only for $X \ll 1$ agrees well with experiment in a much wider interval of variation of X remains a major challenge.

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[1]R. H. Kraichnan, Phys. Fluids **11**, 945 (1968).

[2]S. Corrsin, J. Appl. Phys. **22**, 469 (1951).

[3]E. Ott and T. M. Antonen, Phys. Rev. Lett. **61**, 2839 (1988).

[4]P. G. Saffman, J. Fluid Mech. **16**, 545 (1963).

[5]A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics* (MIT Press, Cambridge, 1975), Vols. 1 and 2, and references therein.

[6]G. K. Batchelor, J. Fluid Mech. **5**, 113 (1959).

[7]R. P. Prasad, C. Meneveau, and K. R. Sreenivasan, Phys. Rev. Lett. **61**, 74 (1988).

[8]C. Meneveau and K. R. Sreenivasan, Nucl. Phys. (Proc. Suppl.) **B2**, 49 (1987).

[9]B. Castaing, G. Gunaratne, F. Heslot, L. Kadanoff, A. Libchaber, S. Thomas, X.-Z. Wu, S. Zaleski, and G. Zanetti, J. Fluid Mech. (to be published).

[10]F. Heslot, B. Castaing, and A. Libchaber, Phys. Rev. A **36**, 5870 (1987).

[11]J. Gollub (private communication).

[12]D. Forster, D. Nelson, and M. J. Stephen, Phys. Rev. A **16**, 732 (1977).

[13]V. Yakhot and S. A. Orszag, Phys. Rev. Lett. **57**, 1722 (1986).

[14]V. Yakhot and S. A. Orszag, J. Sci. Comp. **1**, 3 (1986).

[15]S. B. Pope, Combust. Flame **27**, 299 (1976).

[16]V. Eswaran and S. B. Pope, Phys. Fluids **31**, 506 (1988).

[17]O. Metais and M. Lesieur, in *Proceedings of the Second European Turbulence Conference, August 1988,* edited by H. M. Fernholz and M. E. Fiedler (Springer-Verlag, Berlin, 1989).