Parity-Breaking Transitions of Modulated Patterns in Hydrodynamic Systems

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A model for a subcritical parity-breaking bifurcation of a periodic pattern is introduced. It is found that nucleated regions of the new asymmetric state propagate with a well-defined velocity in a direction determined by their parity, and leave in their wake a pattern with an altered wavelength. Successive passage of these "parity bubbles" enables the systems to relax to a selected wavelength. The possible relevance of these findings to recent observations of "solitary modes" in directional solidification and hydrodynamic experiments is discussed.

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The Rayleigh-Bénard transition of a fluid heated from below[1] and the Mullins-Sekerka[2] instability of a moving interface between two phases are paradigms of a hydrodynamic transition beyond which a uniform pattern in space gives way to a periodic structure. Experiments performed near the onset of these transitions[3] show clearly that the structures which appear (i.e., a pattern of convective rolls or a cellular interface) replace the continuous translational symmetry of the homogeneous system with a discrete one, but that these periodic patterns retain reflection symmetry. Phenomenological approaches to the dynamics of such patterns which invoke only these invariance properties have proved successful on both qualitative and quantitative levels.

Recently, however, it has been observed[4] that beyond the Mullins-Sekerka instability of liquid crystals,[3] there may appear localized, propagating regions within which the pattern is left-right asymmetric, as shown schematically in Fig. 1. These traveling domains possess a number of important properties, notably that they move in a direction determined by the sense of their asymmetry. Analogous observations have been made in other directional solidification experiments,[5] in Rayleigh-Bénard[6] convection, and in studies of a fluid meniscus near a rotating cylinder.[7]

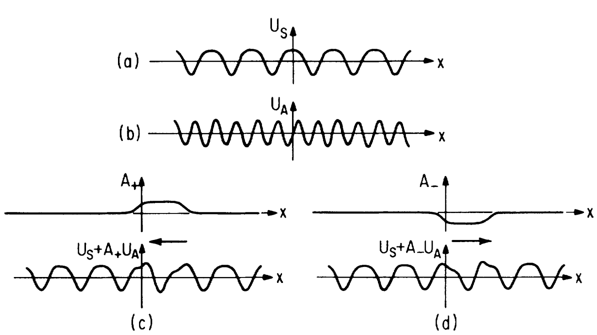

FIG. 1. Resolution of an interface pattern into symmetric and antisymmetric components. (a) The symmetric cellular pattern U_S; (b) a pattern U_A antisymmetric with respect to U_S; (c),(d) two linear combinations of U_A and U_S with localized opposite-parity amplitude functions $A\pm$. Arrows indicate directions of motion.

We suggest that these traveling domains are nucleated inclusions of an *antisymmetric state* whose presence is an indication of a secondary bifurcation to a broken-parity state. On the basis of rather general symmetry considerations, we develop the simplest time-dependent amplitude equations for the transition; the resulting model exhibits many of the qualitative features observed in the experiments.

For concreteness, we cast the following discussion in the language of directional solidification, but expect the arguments to be valid for related experiments. Let us first review the time-dependent Landau-Ginzburg description of the initial Mullins-Sekerka (MS) transition. The planar interface which is stable below the MS instability is, of course, invariant under arbitrary spatial translations $x \rightarrow x+\Sigma$, while beyond the bifurcation the modulated interface possesses the more restricted discrete symmetry $x \rightarrow x+n\lambda$, for any integer n, λ being the fundamental wavelength of the pattern. When the periodic interface just above onset is slightly distorted, it may be described as[8] $U(x)=\mathrm{Re}[C(x)\exp(iq_0x)]$, with $q_0 \equiv 2\pi/\lambda$, and where $C(x)$ is a slowly varying complex amplitude whose dynamics reflects the full translational and reflection symmetries of space. The (gauge-invariant) equation of motion in the normal form is[9] $C_t = C_{xx}+\mathcal{R}(C)$, where $\mathcal{R}$ is a suitable nonlinear operator. Writing $C=A\exp(i\Phi)$ with the amplitude A and phase Φ real, the "up-down" and "left-right" symmetries of the bifurcated state are then manifested in the invariance of the equations of motion for A and Φ under *each* of the transformations $x\rightarrow -x$, $A\rightarrow -A$, and $\Phi\rightarrow -\Phi$, as well as $\Phi\rightarrow\Phi+\text{const.}$

Let us now describe the parity-breaking transition. Imagine an interface function $U(x)$ which is a linear combination of symmetric and antisymmetric profiles U_S and U_A, as shown in Fig. 1;

$$U(x)=SU_S(x+\phi)+AU_A(x+\phi)\,, \qquad (1)$$

with $U_S(z)=U_S(-z)$ and $U_A(z)=-U_A(-z)$, where $S(x)$, $A(x)$, and $\phi(x)$ are slowly varying *real* quantities. The amplitude A serves as the order parameter of the antisymmetry, vanishing prior to the secondary instability. Likewise, lateral translations of the pattern are described by shifts in the phase variable ϕ. The theory outlined below makes no detailed statements concerning the functional forms of U_S and U_A. [For the experiment of Ref. 4, the symmetric pattern is well described as $U_S=a\cos(q_0x)+b\cos(2q_0x)$, with $b/a\simeq -0.25$, as shown in Fig. 1(a)]. The internal structure of the experimentally observed domains suggests that the antisymmetric function U_A has a significant projection on the function $\sin(2q_0x)$, as indicated schematically in Fig. 1(b). With the localized amplitude functions $A\pm$ of opposite parity shown in the figure, we may construct the model interface patterns shown in Figs. 1(c) and 1(d).

The symmetries of the equations of motion may be deduced from the invariance of the dynamics seen by observers on opposite sides of the plane of the pattern. Let $U(x)$ and $\bar{U}(\bar{x})$ be the interface as seen by the two. The lateral coordinates of the viewers are clearly related by $\bar{x}=-x$, so a shift $\phi(x)$ for one corresponds to $-\bar{\phi}(\bar{x})$ for the other. The vertical displacements seen by the two must be identical, and hence (with the antisymmetry of U_A), the amplitudes must be related by $\bar{A}(\bar{x})=-A(x)$. To summarize, we require the covariance[10]

$$\bar{A}(-x)=-A(x)\,,\quad \bar{\phi}(-x)=-\phi(x)\,;$$

$$\partial_t\bar{A}(-x)=-\partial_tA(x)\,,\quad \partial_t\bar{\phi}(-x)=-\partial_t\phi(x)\,,$$

with the latter relationships following directly from the motion seen by the observers.

For slowly varying A and ϕ, the lowest-order equations of motion obeying these symmetries, as well as the invariance under $\phi\to\phi+\text{const}$, are

$$A_t=A_{xx}+f(A)+\epsilon\phi_xA+\cdots\,, \qquad (2)$$

$$\phi_t=\phi_{xx}+\delta A+\cdots\,, \qquad (3)$$

where $f(A)$ is an odd polynomial in A, ϵ and δ are coupling parameters, and we have suppressed unimportant numerical prefactors. Terms such as ϕ_{xx} and AA_x in (2) and A_{xx} in (3) are allowed by symmetry, but are not central to the discussion here.[11]

In light of the nucleation effects seen in experiment, we investigate the form of f appropriate to a first-order phase transition, namely

$$f(A)=\mu A+\alpha A^3-A^5\,. \qquad (4)$$

When $\epsilon=\delta=0$, the diffusive and polynomial contributions in (2) arise from the relaxational dynamics $A_t=-\delta\mathcal{F}/\delta A$, with the free energy $\mathcal{F}=\frac{1}{2}A_x^2+F(A)$, where $-\partial F/\partial A=f(A)$. Thus, $F(A)=-\frac{1}{2}\mu A^2-\frac{1}{4}\alpha A^4+\frac{1}{6}A^6$ is appropriate to a system near a tricritical point, but with $\alpha>0$ exhibiting a first-order transition from the state $A=0$ for $\mu<\mu^*$ to $A=\pm A^*$ for $\mu>\mu^*$, $\mu^*=-\frac{3}{16}\alpha^2$ being the coexistence point at which the three minima of $F(A)$ are equal. As is usual in such time-dependent Landau-Ginzburg models, we assume that sufficiently near the instability the important variation of the Landau coefficients in (2) and (3) resides only in the control parameter μ. For solidification, $\mu\sim(v-v_0)/v_0$, v_0 being some characteristic velocity. We shall see below that the couplings between A and ϕ, whose form is a direct consequence of the antisymmetry of U_A with respect to U_S, provide the driving force for the propagation of domains of antisymmetry ("bubbles").

Numerical and analytical study of the model in (2) and (3) yields a number of qualitative results that are in accord with experiments on directional solidification.[11] First, postulating the existenced of a *transition* to a broken-symmetry state explains the appearance of a lower limit of interface velocities for the existence of the propagating inclusions.[4] Second, we find that the direction of their propagation is determined completely by the parity of the inclusion. Third, the periodic pattern left after the passage of an inclusion has a phase ϕ which is linear in x; i.e., the underlying symmetric pattern has an altered wavelength. The direction of that shift is determined by its parity. Hence, the sign of the wavelength shift is related to the direction of propagation of the inclusion, in accord with experimental observations.[4]

In order to understand the motion of inclusions, it is perhaps simplest first to consider the motion of an elementary *kink*, the interface between a region with $A=0$ and another with $A=A^*$. The kink profile $A_K(x)$ is stable at the coexistence point $\mu=\mu^*$ when $\epsilon=\delta=0$, and satisfies $A_t=0$, with the trivial phase function $\phi_{xx}=0$. There are four such solutions which are uniquely identified by their parity $(\pm)$ and left-right orientation (L,R) as shown in Fig. 2. For small deviations $\Delta\mu$ away from μ^*, each of these kinks will move in a direction determined by the relative stability of the states $A=0$ and $A=\pm A^*$, as shown by the dashed arrows in the figure. The shape and velocity of propagation u of these kinks may be determined as follows. Taking, for example, the "$+L$" kink, we write $A(x,t)=A_L^+(z)+W(z)$, where $z=x-x_0(t)$ and W is a small correction, $u=\dot{x}_0(t)$ being the as yet unknown velocity. Substituting this trial solution into the equations of motion, we arrive at a standard problem in Poincaré perturbation theory involving a "solvability condition" to eliminate secular terms.[12] This lowest-order approximation to the kink velocity is

$$u\simeq-\Delta\mu\frac{\int dz\,A_K(\partial A_K/\partial z)}{\int dz\,(\partial A_K/\partial z)^2}\,. \qquad (5)$$

Inspection of this result shows that *the driving force for*

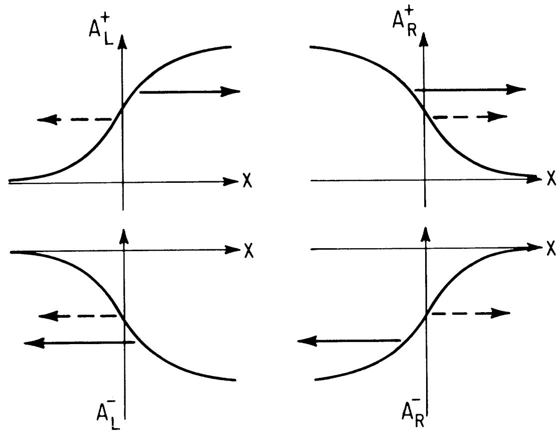

FIG. 2. Four elementary kink structures arising from the amplitude equations. Dashed arrows indicate the directions of propagation due to positive deviations $\mu-\mu^*$; solid arrows indicate the direction of motion arising from amplitude and phase couplings, with $\epsilon<0$ and $\delta>0$.

kink motion arising from the control parameter μ depends only on the kink orientation and not on its parity (see Fig. 2).

Turning now to the role of the terms coupling A and ϕ, we find it convenient to solve Eq. (3) formally for $\psi\equiv\phi_x$ as $\psi=\delta G^{-1}A_x$, G being the Green's function for the diffusion operator $\partial_t-\partial_{xx}$. Substitution back into (2) yields an integro-differential equation for A with the term $A\psi$ now replaced by one with the same symmetry as the quantity AA_x, since G is of unique sign. For the purpose of establishing the role of parity in the motion induced by the couplings, we thus consider the solvability condition appropriate to this term, and find a velocity with the same parity and orientational symmetries as

$$\frac{\int dz\, A_K(\partial A_K/\partial z)^2}{\int dz(\partial A_K/\partial z)^2}, \tag{6}$$

which is manifestly antisymmetric under parity inversion and symmetric under orientational inversion. Thus, *the couplings between amplitude and phase lead to motion depending only on parity and not on orientation* (Fig. 2).

Now we may join in pairs elementary propagating kinks of opposite orientation and similar parity to create *propagating bubbles*. As a consequence of the above, we see immediately that the propagation velocities v_l and v_r of the left and right edges of the inclusion are not equal, the difference being directly related to the deviation of μ from the coexistence point; that is, the inclusions can grow or shrink in time, as has been observed in recent experiments.[4] Figure 3 shows the time evolution of the amplitude and phase of a propagating inclusion, as obtained from a numerical solution of (2) and (3). We find nucleationlike effects in that a localized disturbance around the homogeneous state only leads to a propagating bubble if it exceeds some characteristic size.

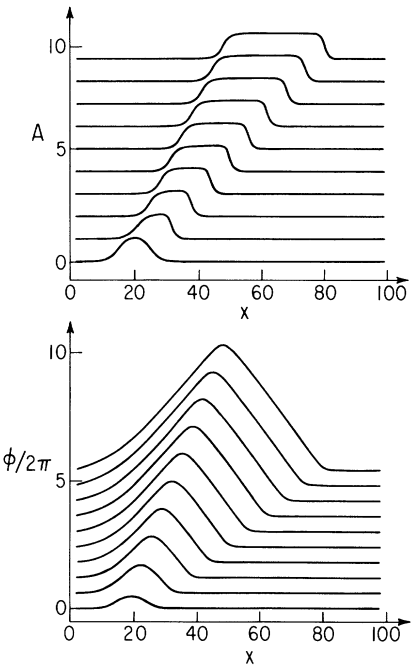

FIG. 3. Time evolution of amplitude and phase functions, as obtained by numerical solution of Eqs. (2) and (3), with $\mu=-0.2$, $\alpha=\delta=1.0$, and $\epsilon=-1.0$. Successive snapshots have been displaced vertically for clarity; the linear phase left in the wake of the bubble actually overlaps at different times.

After the passage of the bubble, we find that the local phase function left behind is linear; $\phi=\mathrm{sgn}(\epsilon)qx$, with $q\propto\Delta\mu$. This is equivalent to a shift in the wave vector of the underlying symmetric pattern by an amount $\pm q$. (Observe that a linear phase function and vanishing amplitude are static solutions to the model dynamics.) It is observed experimentally that spreading bubbles leave behind a pattern of shorter wavelength; this implies $\epsilon<0$. A relation exists between the wave vector chosen by the system and the rate of spreading of the bubble, as may be seen by approximating the shape of the inclusion by a square wave whose left and right edges move at velocities v_l and v_r; $A(x,t)\simeq A^*\Theta(x-v_lt)\Theta(v_rt-x)$, with Θ being the Heaviside function. Neglecting the diffusive term in (3), we may simply integrate in time to obtain

$$\phi(x,t)\simeq\begin{cases}\delta A^*(v_r-v_l)x/v_lv_r, & 0<x<v_lt,\\ \delta A^*(t-x/v_r), & v_lt<x<v_rt\,,\end{cases} \tag{7}$$

with $\phi=0$ for $x\leq 0$ and $x\geq v_rt$. The triangular shape of the phase function (7) is in essential agreement with the full numerical solution in Fig. 3. Of course, the detailed motion of an inclusion is also affected by interactions between its front and back edges, but this exponentially decaying coupling is quantitatively important only for small inclusions.

The relation between the spreading rate of a moving inclusion of the broken-parity state and the wave-vector shift left in its wake has an interesting consequence for pattern selection. Note that when the wave-vector shift is $\phi_x = q$, the effective control parameter, that is, the coefficient of the linear term in A in Eq. (2), is $\mu + \epsilon q$. Thus, with $\epsilon < 0$ and q having the sign of $\mu - \mu^*$, we see that repeated passage of bubbles tends to drive μ to μ^*; *the system relaxes to a selected wavelength by the successive passage of parity bubbles.*

In conclusion, we find that several qualitative features of pattern-forming systems exhibiting localized asymmetric modes follow directly from a simple time-dependent Landau-Ginzburg model of a dynamical transition to a broken-symmetry state. This phenomenon of parity breaking appears to be associated with a fundamentally new mechanism for wavelength selection in certain hydrodynamic systems. The generality of the derivation of the model leads us to believe that there should be other classes of systems in which propagating inclusions of broken-parity states will appear. Future work in this area will focus on higher-dimensional generalizations of this transition, the dynamics of defect structures, and an understanding of the microscopic origins of parity-breaking bifurcations in the diverse experimental systems.

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