Monopoles and Confinement in Three Dimensions

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The string tension is calculated using magnetic monopoles in three-dimensional U(1) lattice gauge theory. The monopoles are identified in configurations of link angles generated in a simulation on a 32^3 lattice. Wilson-loop values are determined by the monopoles' flux through the loop. The string-tension results are in excellent agreement with a semiclassical analytic formula.

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How is the topology of the vacuum related to confinement? A quantitative answer to this question has been elusive for lattice QCD. In this paper, we report on numerical calculations for the case of U(1) lattice gauge theory in three dimensions. Our principal result is a calculation of the string tension which directly involves magnetic monopoles.

The physics studied here was first discussed by Polyakov[1,2] who considered a three-dimensional continuum theory in which the essential feature was an unbroken U(1) gauge group with monopoles as instantons. Electric confinement was established by a semiclassical analysis of the multimonopole gas. In an analogous lattice treatment, Banks, Myerson, and Kogut derived an explicit formula for the string tension in Villain's version of the U(1) theory.[3] Our simulations establish that this formula for the string tension is accurate over an unexpectedly wide range of couplings.

To begin our calculations, values of Wilson loops were gathered on a 32^3 lattice.[4] Because it is efficient to simulate, we used the cosine form of the U(1) action:[5]

$$S=\beta_W \sum_{\mathbf{x},\mu>\nu} [1-\cos\theta_{\mu\nu}(\mathbf{x})] , \qquad (1)$$

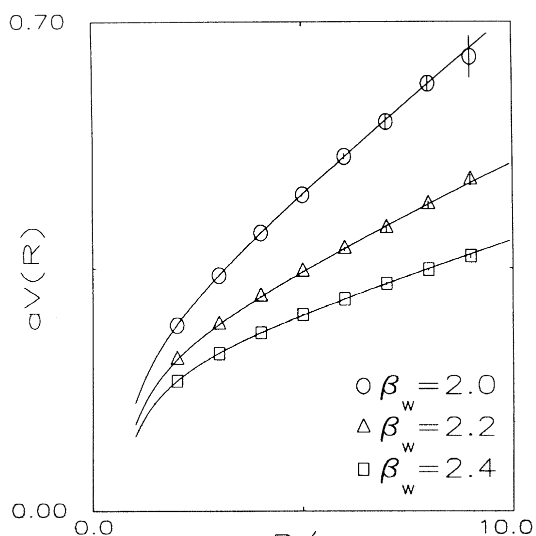

FIG. 1. The potentials from the original Wilson loops and their linear-plus-Coulomb fits.

where

$$\theta_{\mu\nu}(\mathbf{x})=\theta_\mu(\mathbf{x})+\theta_\nu(\mathbf{x}+\hat{\mu}a)-\theta_\mu(\mathbf{x}+\hat{\nu}a)-\theta_\nu(\mathbf{x}) ,$$

and a is the lattice spacing. A heat-bath algorithm was used to update the links.[6] Wilson-loop statistics were enhanced with an analytic version of the multihit procedure of Parisi, Petronzio, and Rapuano.[7] At $\beta_W=2.0$, 2.2, and 2.4, we made runs of 25000 sweeps, dropped the first 5000, and measured all Wilson loops up to $R=10a\times T=14a$, measuring every 25 sweeps. At these β_W values, the string tension σ defines a correlation length $\xi=1/\sqrt{\sigma}$ which lies approximately between four and eight lattice spacings. This range of correlation lengths is large enough for continuum behavior to be seen, but is still considerably smaller than the lattice size of $32a$.

From the Wilson-loop values $W(R,T)$, the potentials $V(R)$ were determined by fits of $\ln W(R,T)$ vs T.[8] Then we extracted the string tension σ at each β_W by performing linear-plus-Coulomb fits to the potentials: $V(R)=-\alpha/R+\sigma R+V_0$. The results are shown in Fig. 1 and Table I (see the column labeled "original").

We now ask, can the same results be obtained using the monopole gas of Polyakov? This means extracting a set of monopole locations $\{m(\mathbf{x})\}$ from the configurations of link angles, then using these monopole locations to form a second estimate of the Wilson loops and the string tension.

DeGrand and Toussaint introduced a procedure which locates a monopole by finding the end of its Dirac string.[9] First, every plaquette in a configuration is represented as $\theta_{\mu\nu}(\mathbf{x})=2\pi n_{\mu\nu}(\mathbf{x})+\theta'_{\mu\nu}(\mathbf{x})$, where $n_{\mu\nu}(\mathbf{x})$ is an integer, and $\theta'_{\mu\nu}(\mathbf{x})$ is in the range $[-\pi,\pi]$. The $n_{\mu\nu}$ are

TABLE I. Results of fits with $V(R)=-\alpha/R+\sigma R+V_0$ for original and monopole potentials. The columns β_W and β_V refer to the parameter in the cosine and Villain actions, respectively.

		Original		Monopole	
β_W	β_V	σ	α	σ	α
2.0	1.389	0.049(3)	0.13(4)	0.048(1)	0.22(2)
2.2	1.575	0.029(2)	0.13(8)	0.029(1)	0.22(3)
2.4	1.768	0.017(1)	0.17(5)	0.017(1)	0.25(4)

then used to define a net flux emerging from each elementary cube, and a monopole with charge equal to this flux is assigned to the cube center. At the values of β_W we use, only charges 0 and ± 1 actually occur, although the procedure allows 0, ± 1 and ± 2 to be generated. The average number of monopoles found in a configuration ranged from ~ 200 at $\beta_W = 2.0$ to ~ 20 at $\beta_W = 2.4$.

In calculating Wilson loops in terms of monopoles, our basic assumption is that the gauge fields can be split into a contribution from monopoles plus free photons, i.e., $\mathbf{A} = \mathbf{A}_{\rm mon} + \mathbf{A}_{\rm ph}$.[10] This leads to a factored form for $W(R,T)$:

$$W(R,T) = W_{\rm ph}(R,T) W_{\rm mon}(R,T) . \tag{2}$$

The factor $W_{\rm ph}$ can be evaluated as a standard Gaussian path integral:

$$W_{\rm ph} = \left\langle \exp\left[i \sum \mathbf{A}_{\rm ph} \cdot \mathbf{J} \right] \right\rangle = \exp\left[-\frac{e^2}{2} \sum_{\mathbf{x},\mathbf{x}'} \mathbf{J}(\mathbf{x}) v_c(\mathbf{x}-\mathbf{x}') \cdot \mathbf{J}(\mathbf{x}') \right] , \tag{3}$$

where $\mathbf{J}(\mathbf{x})$ is the integer-valued current defining the Wilson loop, and $v_c(\mathbf{x}-\mathbf{x}')$ is the lattice Coulomb potential. In the weak-coupling limit, $\beta_W \to \infty$ and $e^2 \to 1/\beta_W$. However, at the finite values of β_W used here, it is essential that we do not set $e^2 = 1/\beta_W$, but allow a more general dependence on β_W.

The monopole contribution to a Wilson loop may be written

$$W_{\rm mon} = \left\langle \exp\left[i \sum \mathbf{A}_{\rm mon} \cdot \mathbf{J} \right] \right\rangle = \left\langle \exp\left[-2\pi i \sum_{\mathbf{x}} \nabla\phi_m(\mathbf{x}) \cdot \mathbf{D}(\mathbf{x}) \right] \right\rangle , \tag{4}$$

where the second equality in Eq. (4) follows if we set $\mathbf{J} = \nabla \times \mathbf{D}$, and perform the lattice analog of integration by parts. The magnetic field of the monopoles can be represented as $-2\pi\nabla\phi_m$ since the Dirac strings do not contribute in the exponent. For a set of monopole charges $m(\mathbf{x})$, the scalar potential $\phi_m(\mathbf{x})$ is given by

$$\phi_m(\mathbf{x}) = \sum_{\mathbf{x}'} v_c(\mathbf{x}-\mathbf{x}') m(\mathbf{x}') . \tag{5}$$

The dipole density $\mathbf{D}$ is defined on an arbitrary surface of plaquettes with boundary $\mathbf{J}$. The argument of the exponent in Eq. (4) is independent of the surface chosen, and represents the magnetic flux due to the monopoles through the Wilson loop. For convenience, the surface defining $\mathbf{D}$ was taken to be in the loop plane.

We now turn to a discussion of the assumption Eq. (2), and the proper choice of $e^2(\beta_W)$ in $W_{\rm ph}$. In Villain's form of the U(1) theory, Eqs. (2)-(4) are exact, with $e^2 = 1/\beta_V$.[3] The parameter β_V appears in the Villain action as follows:

$$\exp(-S_V) = \prod_{\mathbf{x},\mu>\nu} \sum_{m_{\mu\nu}(\mathbf{x}) = -\infty} \exp\left[-\frac{\beta_V}{2} [\theta_{\mu\nu}(\mathbf{x}) - m_{\mu\nu}(\mathbf{x})]^2 \right] . \tag{6}$$

According to universality, the same values for large Wilson loops (up to a nonuniversal perimeter term) can be obtained using either the cosine action at β_W or the Villain action at β_V, provided the system is close enough to the continuum limit. The relation between β_V and β_W would be determined in principle by matching correlation lengths ξ, or equivalently, string tensions:

$$\sigma(\beta_V) = \sigma(\beta_W) . \tag{7}$$

We assume the way universality is realized is that Eq. (2), already exact for the Villain action, becomes accurate for the cosine action as we approach the continuum limit. Since the photons are free and do not interact with the monopoles, it follows that $W_{\rm ph}$ from a cosine-action calculation at β_W must equal $W_{\rm ph}$ from a Villain calculation at β_V. This will be satisfied only if in $W_{\rm ph}$ from the cosine action, $e^2(\beta_W)$ is evaluated as $1/\beta_V(\beta_W)$.

The above discussion leaves unanswered the question of whether or not universal behavior can be expected at our values of β_W. To investigate this, we directly simulated the Villain and cosine actions and compared results. These runs were shorter and on smaller lattices than those on the 32^3 lattice. It turned out to be unnecessary to actually use Eq. (7) to find $\beta_V(\beta_W)$. In his original paper, Villain[11] determined an approximate form of the function $\beta_V(\beta_W)$:

$$\beta_V(\beta_W) = \left[2 \ln\left[\frac{I_0(\beta_W)}{I_1(\beta_W)} \right] \right]^{-1} , \tag{8}$$

where I_0 and I_1 are the standard modified Bessel functions. We found this formula to be quite accurate. Using it to relate β_V and β_W, Wilson loops from Villain- and cosine-action simulations differed only by a perimeter term, to within statistical errors. These results are consistent with universality. The values of β_V corresponding to our values of β_W are given in Table I. The asymptotic relation $\beta_V \approx \beta_W$ is clearly not satisfied.

Before turning to the actual use of Eq. (2), we discuss a subtle point involving the monopole identification scheme. In the Villain model, the path integral using the action of Eq. (6) can be transformed via Fourier series and the Poisson-sum formula into a Coulomb-gas representation, where the $\theta_\mu(\mathbf{x})$ no longer appear. The basic variables in the gas representation are the monopole locations $m(\mathbf{x})$. Free photons contribute an overall perturbative factor [cf. Eq. (9) of Banks, Myerson, and Kogut[3]]. It is of interest to confirm that the monopoles we identify in the gauge-field configurations are physically the same objects as those of the gas representation. On a

10^3 lattice, we performed two independent simulations of the Villain model: a conventional simulation for the action of Eq. (6), and a direct simulation of the Coulomb gas. The output of these simulations was two sets of monopole locations: one from the gauge-field configurations of the conventional simulation, and the other from the gas simulation. For $\beta_V \geq 0.6$, the two distributions produced identical results for the total Coulomb energy of the monopoles,[12] and for Wilson loops using Eq. (2), to within statistical errors. Thus for the couplings of interest here, the DeGrand-Toussaint scheme does correctly identify the monopoles in the gauge-field configurations.[13]

Having verified that the monopoles are being correctly identified, and that there is universality between cosine and Villain actions, we proceed to use Eq. (2) to evaluate Wilson loops from our 32^3 cosine-action configurations. It is straightforward to calculate $W_{\rm ph}$, where, by the discussion above, e^2 is to be evaluated as $1/\beta_V(\beta_W)$, using Eq. (8) for $\beta_V(\beta_W)$. The monopole locations are extracted from the configurations, and used in Eq. (5) to calculate ϕ_m. Then $W_{\rm mon}$ is calculated using Eq. (4), where the expected value implies an average over configurations. Since the monopoles evolve very slowly in the course of a run, configurations used in this average were spaced by 200 sweeps. Combining $W_{\rm ph}$ and $W_{\rm mon}$ finally results in a second estimate of Wilson loops which we call "monopole" loops in the following.

The potentials from the monopole loops are shown in Fig. 2, along with the potentials from the original Wilson loops. A constant has been removed from the monopole potentials at each β_W. This constant is a reflection of the perimeter term mentioned previously. To avoid crowding of points in the vertical direction, the original and monopole potentials at $\beta_W = 2.0$ and 2.2 have been shifted upward by 0.2 and 0.1, respectively. The $\beta_W = 2.4$ potentials have been left unshifted. Being independent of R, neither the removal of the constant nor the upward shift affects any physical quantity.

From Fig. 2, it is clear that for large R, the potentials agree. For small R, the monopole potentials have slightly greater curvature. This small-R discrepancy is not surprising. Our method of calculation of the Wilson loops from monopoles relies on assumptions about universality. Universality applies on the scale of the correlation length and greater; nonuniversal behavior can still be present on the scale of the lattice spacing.

In Fig. 3 we show the potential and the separate contributions from photons and monopoles, for $\beta_W = 2.0$. It is clear that the strong rise in the potential for large R is due to the monopoles. This gives graphic evidence that the nonperturbative effects caused by monopoles are ultimately responsible for confinement.

We performed linear-plus-Coulomb fits to the monopole potentials. The results are presented in Table I. The extra curvature present in the monopole potentials at small R shows itself in high values for α. The value obtained for α is completely controlled by the smallest R values included in the fit. In contrast, the value obtained for the string tension is controlled by the large-R region. To within statistical errors, the string tension calculated from the monopole loops agrees with the string tension obtained from the original Wilson loops at each value of β_W. This is our main result and shows that our treatment of the monopoles and our handling of universality are self-consistent at large R.

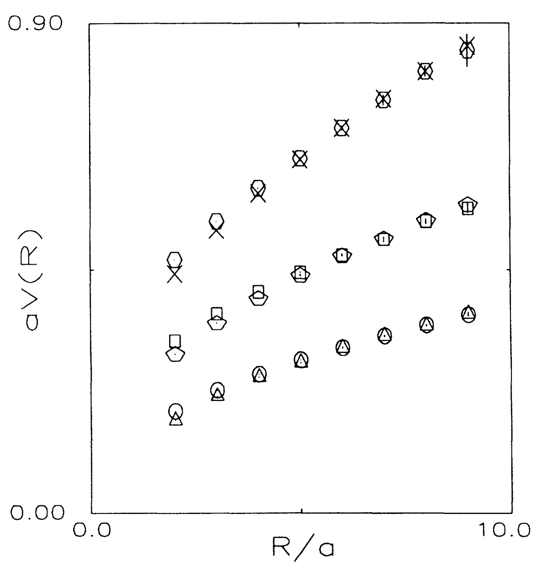

FIG. 2. Comparison of the potentials from the original Wilson loops (hexagons, $\beta_W = 2.0$; squares, $\beta_W = 2.2$; circles, $\beta_W = 2.4$) with the potentials from monopole loops (crosses, $\beta_W = 2.0$; pentagons, $\beta_W = 2.2$; triangles, $\beta_W = 2.4$).

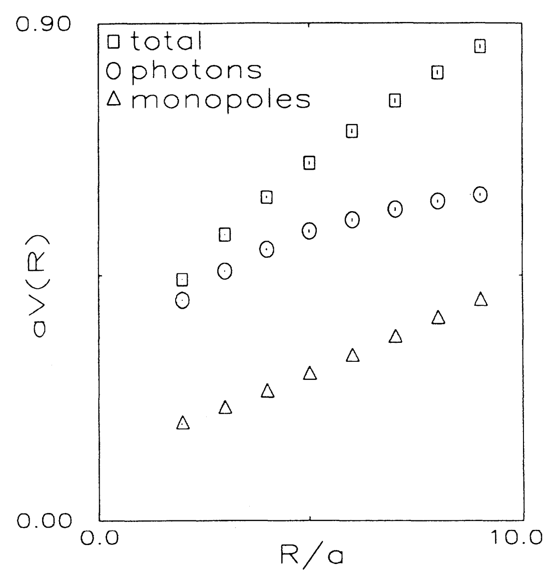

FIG. 3. The contributions to the potential from photons and monopoles for $\beta_W = 2.0$.

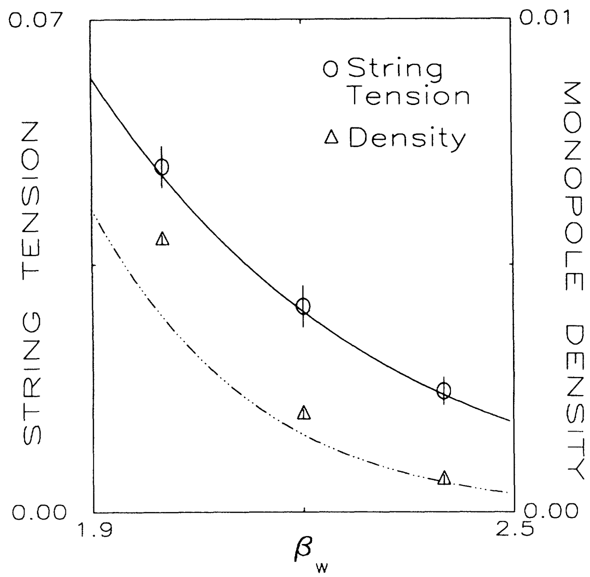

FIG. 4. Upper curve: Measured values of string tension vs Eq. (9). Lower curve: Measured values of the monopole density vs $\pi^2\beta_V\sigma^2/8$.

While it is no easier computationally, getting the string tension from the monopoles is more revealing than the usual method in that it is directly tied to the vacuum topology and the mechanism of confinement. Although monopoles are explicit only in the Coulomb-gas form of the Villain model, it is satisfying that the physics they describe can be extracted directly from the gauge fields, and indeed using a different action. We have used the cosine form of the action, but since universality is well satisfied, any reasonable periodic action could have been used. Our results for three-dimensional U(1) show that quantitative lattice-gauge-theory calculations with topological objects are possible. Although much more difficult, similar calculations should be possible in the monopole approach to confinement in lattice QCD.[14]

Finally, we compare our results with analytic formulas derived semiclassically. The authors of Ref. 3 (see also Ref. 15), using the Villain action, derived the following for the string tension:

$$\sigma(\beta_V)=\frac{4\sqrt{2}}{\pi\sqrt{\beta_V}}\exp[-\pi^2 v_c(0)\beta_V]\,, \tag{9}$$

where $v_c(0)=0.2527\ldots$. In Fig. 4, we plot our results for σ versus the string tension from Eq. (9), using Eq. (8) to compute $\beta_V(\beta_W)$. The agreement is remarkable.[16] We have also measured the total density of monopoles, n. Expressed in terms of σ, the monopole density is given by $n=\pi^2\beta_V\sigma^2/8$. In the same figure, we show our results for n versus this formula, again using Eq. (9) for σ and Eq. (8) for $\beta_V(\beta_W)$. The measured values of n approach the theoretical curve from above, reaching agreement at $\beta_W=2.4$. It appears quite likely that beyond $\beta_W=2.4$, both σ and n will agree with the theoretical formulas. If so, then these formulas hold at surprisingly small β_W. The expected range of validity for the theoretical results involves the Debye screening length λ_D, related to σ by $\lambda_D=2/\pi^2\beta_V\sigma$. Polyakov gives $n\lambda_D^3\gg 1$ as the condition for validity of the semiclassical analysis.[1] However, $n\lambda_D^3$ is only ≈ 0.2 at $\beta_W=2.4$, and $n\lambda_D^3\gtrsim 1$ requires $\beta_W\gtrsim 3.3$, or correlation lengths $\xi \gtrsim 28a$. The results of our simulation suggest that while the condition $n\lambda_D^3\gg 1$ is certainly sufficient, it is overly restrictive, and that the semiclassical analytic results hold over a considerably wider range of couplings.

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