Griffiths Singularities in Two-Dimensional Random-Bond Ising Models: Relation with Lifshitz Band Tails

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Ising models with correlated or uncorrelated random horizontal bonds are studied for square lattices in zero field. The partition sum is expressed in terms of a one-component field. Griffiths singularities in the free energy as function of temperature are derived. They are logarithmic transforms of certain Lifshitz band tails. Griffiths singularities occur above as well as below the critical temperature.

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Twenty years ago Griffiths[1] proved that the magnetization of a randomly diluted Ising ferromagnet is nonanalytic when the external field h goes to zero, for temperatures smaller than some temperature T_G. This "Griffiths temperature" is larger than the critical temperature T_c of the random system. The Griffiths singularity occurs not only in the diluted Ising model, but quite generally in systems with disorder. In percolation models, e.g., the singularity can be calculated explicitly, see Kunz and Souillard.[2] Griffiths[1] also considered the possibility of nonanalytic behavior in the temperature variable at zero field, but did not reach conclusive results. This will be the topic of the present Letter, for general random Ising magnets on a square lattice.

The Griffiths temperature is equal to the critical temperature of a fictitious pure system, where all couplings take the largest value allowed in the random system. The temperature interval $T_c < T < T_G$ is commonly called the Griffiths phase. The singularity at T_G is caused by large, but improbable, pure regions with strongest couplings. In an infinite system these domains may be arbitrarily large, and, therefore, arbitrarily close to a phase transition at T_G. Since the probability for their occurrence decreases exponentially with their size, the contribution to the free energy is expected to involve an essential singularity.

Because of the mechanism of critical slowing down, the long-time dynamics of random magnets in the Griffiths phase is determined by such clusters.[3-5] Hence what leads to an invisible singularity in the free energy shows up as the only contribution in the long-time dynamics. This is connected to another fundamental phenomenon, namely, the Lifshitz[6] band tail in the density of states of random problems with linear equations of motion, such as tight-binding or harmonic systems. Lifshitz showed that, at band edges of random systems, an exponential singularity replaces the van Hove singularity occurring in pure systems. Consider, for instance, diffusion in an infinite medium with traps at fixed random positions. Here the long-time survival is slower than exponential, because of the presence of arbitrarily large clusters without traps.[7] A recent discussion of this problem for the three-dimensional situation was given by the present author.[8] It was shown that, for small concentration of traps, the asymptotic Lifshitz tail can be extended to an energy interval of more practical interest. In this way the crossover between an initial region with essentially exponential decay and the asymptotic stretched exponential decay[7] in the trapping problem has been determined.

A connection between Lifshitz tails and Griffiths singularities has been suspected for awhile. It is the purpose of the present work to show that both are indeed related to Ising models on a square lattice with random bonds in zero field. The essential point is that the partition sum can be expressed in terms of a *one-component* free fermion field if either horizontal or vertical couplings are random. Given that form, recent results on Lifshitz tails[9,10] can be used to study the singularity of the free energy as a function of temperature in zero field for arbitrary distributions of disorder.

We shall consider two types of models. In the first one, the random horizontal bonds are fully correlated in the vertical direction. In the second model, disorder is uncorrelated. The system with correlated randomness was introduced by McCoy and Wu (MW),[11] who showed that the free energy has an essential singularity at the ferromagnetic phase transition. The calculations of MW were redone by exact methods in Ref. 12; it was found that the MW result is correct up to a prefactor.

Our main results concern the singularity of the free energy at the Griffiths temperature T_G in random bond Ising models. These results are presented next. For comparison we recall that the singular part of the free energy in Onsager's solution for pure systems behaves as

$$-\beta F_{\text{sing}} \sim \int d\vartheta\, d\varphi \ln(t^2+\vartheta^2+\varphi^2) \sim t^2 \ln t^2 , \qquad (1)$$

where ϑ and φ are wave numbers in the vertical and horizontal directions, respectively. Here all numerical factors have been put equal to unity and $t \sim T - T_c$ measures the distance to the critical point. Let us now consider the MW model, where disorder is fully correlated in one direction. In this system translational invariance is still present in one direction, and ϑ will appear as in

(1). However, for obtaining a value $\varphi = \pi/N$, one needs a strip of width N containing only the stronger of the random couplings. This occurs with probability $c^N \equiv \exp(-\lambda N)$, where c is the probability to find a strong bond. Hence there will be a singularity

$$-\beta F_{\text{sing}} \sim \sum_N e^{-\lambda N} \int d\vartheta \ln(t^2+\vartheta^2+\pi^2/N^2)$$
$$\sim \int_0^\infty e^{-x} dx \ln(1+t^2x^2) + \text{const} \qquad (2)$$

for small $t \sim T - T_G$. The perturbation expansion for small t^2 has zero radius of convergence, so (2) is nonanalytic at $t=0$. An expression similar to (2) was derived by Forgacs, Wolff, and Suto,[13] who studied an Ising model with correlated infinite pinning fields, pointing in an up or down direction.

Next, consider uncorrelated disorder. Here we find that the singularity is even weaker [namely, of the order $\exp(-1/t^2)$] than for the correlated case. The reason is that in this case disorder has to be sampled from a two-dimensional distribution. Qualitatively, this may be understood as follows. One needs (in the correct units) spherical regions with only strong couplings. These regions occur with probability $\exp(-\lambda\pi R^2)$ and lead to

$$-\beta F_{\text{sing}} \sim \int e^{-\lambda\pi R^2} dR \ln(t^2+1/R^2)$$
$$\sim \int e^{-x} dx \ln(1+t^2x) + \text{const}. \qquad (3)$$

We shall now substantiate these results by exact analysis. We consider an Ising model on a square lattice with vertical couplings J_1 and random horizontal couplings $J_2(n,m)$. The partition sum can be written as

$$Z = \left\{ \prod_{n,m} 2\cosh\beta J_1 \cosh\beta J_2(n,m) \right\} Z_p , \qquad (4)$$

where Z_p is a sum over closed polygons, which can be expressed in terms of Grassmann variables; see, e.g., Ref. 14. Here we follow the notation of Ref. 12:

$$Z_p = \int D\psi_1 D\psi_2 D\psi_3 D\psi_4 e^{B_4} , \qquad (5)$$

where integrals are performed at each site and where

$$B_4 = \sum_{n,m} [z_1\psi_4(n,m)\psi_2(n,m+1) + z_2(n,m)\psi_3(n-1,m)\psi_1(n,m) + \{\psi_1\psi_2+\psi_3\psi_4+\psi_1\psi_4+\psi_2\psi_3+\psi_4\psi_2+\psi_3\psi_1\}(n,m)] \qquad (6)$$

involves $z_1 = \tanh(\beta J_1)$ and $z_2(n,m) = \tanh[\beta J_2(n,m)]$. We perform the integrals with respect to ψ_2 and ψ_3 and square the result. This leads to independent complex fields $\bar\psi_{1,4}$ and $\psi_{1,4}$. The integrals over $\bar\psi_4$ and ψ_4 can be performed by Fourier transformation in the m direction at fixed n, because J_1 is nonrandom. The result is, with ψ denoting ψ_1,

$$Z_p^2 = \left\{ \prod_{n,\vartheta} (-2iz_1 \sin\vartheta) \right\} \int D\bar\psi D\psi e^{B_1} , \qquad (7)$$

where $\vartheta = 2\pi j/M$ with $-M/2 < j \le M/2$ for a system of size NM with periodic boundary conditions, and where

$$B_1 = \sum_{\substack{n,m\\ \vartheta,m'}} \frac{e^{i\vartheta(m-m')}}{-2iMz_1\sin\vartheta} \{ -z_2(n,m)(1-z_1^2)\bar\psi(n,m)\psi(n-1,m') - z_2(n,m')(1-z_1^2)\bar\psi(n-1,m)\psi(n,m')$$
$$+ (z_1^2+1-2z_1\cos\vartheta)\bar\psi(n,m)\psi(n,m') + z_2(n,m)z_2(n,m')(z_1^2+1+2z_1\cos\vartheta)\bar\psi(n,m)\psi(n,m') \} . \qquad (8)$$

In the MW model one chooses z_2 independent of m and (8) is diagonalized by introducing Fourier transforms of $\bar\psi$ and ψ at fixed n.[12] If disorder is uncorrelated, it appears in a nonlocal way in (8). The trick to avoid this is as follows. The integral (7) may be written as $\det(CDE)$, where the matrix $D_{n\vartheta;n'\vartheta'}$ is diagonal with elements $(z_1^2+1+2z_1\cos\vartheta)/(-2iz_1\sin\vartheta)$. This matrix is taken apart, and $\mathcal{H}$ is defined as the product CE. For large M, the determinant of D simplifies and Eq. (7) becomes $Z_p^2 = \det\mathcal{H}$, where

$$(\mathcal{H}\psi)(n,m) = [z_2(n,m)^2 - 1]\psi(n,m)$$
$$+ \sum_{m'} (-z_1)^{|m-m'|} \left\{ 2\frac{1+z_1^2}{1-z_1^2}\psi(n,m') - z_2(n,m)\psi(n-1,m') - z_2(n+1,m')\psi(n+1,m') \right\} , \qquad (9)$$

in which disorder occurs only locally. The free energy becomes

$$-\beta F = \langle \ln(2\cosh\beta J_1 \cosh\beta J_2) \rangle + \tfrac{1}{2} \int \rho(E) dE \ln E , \qquad (10)$$

where $\rho(E)$ is the density of eigenvalues of the matrix $\mathcal{H}$. The result (9) and (10) is exact and fully general. It could be used, for instance, to study the ferromagnetic phase transition. A promising route seems to be a supersymmetric description either for $\rho(E)$ (Ref. 10) or for the internal energy. As compared to usual approaches,[14,15] we note that (9) involves only a one-component field. This is related to the fact that we only take the horizontal couplings to be random..

The integral in Eq. (10) will have a singularity when an energy gap at $E=0$ closes. This will yield the Griffiths singularities in zero field. We therefore investigate the *Lifshitz band tail* in the density of states $\rho(E)$ at $E=0$, restricting ourselves to discrete distributions of disorder. We shall assume that each random coupling

$J_2(n,m)$ takes one out of k $(k \geq 2)$ different values $J_{2,j}$ $(J_{2,1} < J_{2,2} < \cdots < J_{2,k})$ with probability $c_j \equiv \exp(-\lambda_j)$, where $\sum_{j=1}^{k} c_j = 1$. The critical temperature of a pure system with all horizontal couplings equal to $J_{2,j}$ will be denoted by $T_{c,j}$ $(T_{c,1} < T_{c,2} < \cdots < T_{c,k} \equiv T_G)$. Further we shall use the notation $\zeta \equiv \tanh(\beta J_1)$ and $z_j = \tanh(\beta J_{2,j})$.

We first study the case of correlated disorder, where $J_2(n,m) = J_2(n)$. Here a Fourier transformation in the m direction introduces an angle ϑ. For general systems the leading Lifshitz band tail in $\rho(E)$ near $E=0$ was derived in Ref. 10. Applying the method here, we find

$$\rho(E) \sim \exp[-A(E)], \tag{11}$$

where $A(E)$ is the minimal value of the effective "action" $-\ln\langle \exp[-\varphi \mathcal{H} \varphi + E\varphi\varphi] \rangle$. In the present situation it becomes

$$A(E) = \sum_n \left\{ (\eta^2 \vartheta^2 - E)\varphi_n^2 - \ln\left[\sum_{j=1}^{k} c_j e^{-(z_j \varphi_n - \zeta^* \varphi_{n-1})^2} \right] \right\}. \tag{12}$$

Here $\eta = 2\zeta/(1+\zeta)^2$ and $\zeta^* = (1-\zeta)/(1+\zeta)$, the dual of ζ (the have been approximated by their $\vartheta = 0$ values). Minimalization of $A(E)$ leads to a "classical" equation of motion for the "instanton" φ_n.

Let us first consider T close to $T_{c,i}$ $(i = 1,2,\ldots,k)$ and assume that E and ϑ are small. Then $t \equiv z_i - \zeta^*$ is small and we can go to the continuum limit where $\varphi_n = \varphi(n)$ is slowly varying. To leading order, Eq. (12) becomes

$$A(t^2 + \eta^2\vartheta^2 + \xi^2\epsilon) = \int_{-\infty}^{\infty} dx \left\{ \lambda_i - \xi^2 \varphi(\epsilon\varphi + \varphi'') - \ln\left[1 + \sum_{j \neq i} \frac{c_j}{c_i} e^{-(z_j - \zeta^*)^2 \varphi^2} \right] \right\}, \tag{13}$$

where $\xi \approx \zeta^*$. It was noted by Lubensky[16] that $\varphi(x)$ is large in a large range of x values. The logarithm can be neglected there and $\varphi(x) = \cos(x\sqrt{\epsilon})/\sqrt{\epsilon}$ gives the minimum. Near $|x| \sim \pi/2\sqrt{\epsilon}$ the logarithm sets in and brings an exponential decay of φ. The leading value of A comes from the former region, where the integrand equals λ_i. Thus ρ behaves as $\exp(-\pi\lambda_i/\sqrt{\epsilon})$ for small ϵ. A more complete description of Lifshitz tails in one dimension was given in Refs. 17 and 18 by analysis of exact Dyson-Schmidt integral equations. The full result for the integrand density H will have the form

$$H(\epsilon) = \exp(-\pi\lambda_i/\sqrt{\epsilon}) R(\pi/\sqrt{\epsilon}) \quad (\epsilon \to 0^+), \tag{14}$$

where R is a periodic function with unit period. This result has to be inserted in (10), which leads to

$$-\beta F_{\text{sing}} \approx (4\pi)^{-1} \int\int d\vartheta\, dH(\epsilon) \ln(t^2 + \eta^2\vartheta^2 + \xi^2\epsilon) \tag{15}$$

in accordance with (2). The results (2), (14), and (15) show two important aspects. First, like in the pure system, the singularity is symmetric in temperature. Second, we obtain a Griffiths singularity near each $T_{c,j}$. The largest of them is, by definition, T_G. Note, however, that (at least) the smallest of them lies below T_c: Griffiths singularities occur as well below the critical temperature of the random system.[18] The difference with ordering of clusters above T_c is that their surface is essentially ordered; their bulk, however, is not yet ordered, and this brings the singularity.

A very similar singularity is present in the random system near the critical point of any pure system having a periodic arrangement of allowed couplings. As an example we consider the behavior near the critical temperature $T_{c,ij}$ corresponding to a pure system where $J_2(2n) = J_i$ and $J_2(2n+1) = J_j$. In this case the dominant term in the logarithm in (12) involves z_i for n even and z_j for n odd. In the region where φ is large, it can be eliminated at the odd sites in favor of φ at even sites. This leads to a similar problem as above, with

$$\xi^2 = 4\zeta^{*2} z_i z_j / (\zeta^{*2} + z_i^2 + z_j^2)$$

and

$$t = (\zeta^{*2} - z_i z_j)/(\zeta^{*2} + z_i^2 + z_j^2)^{1/2} \sim T - T_{c,ij}.$$

Further $(\lambda_i + \lambda_j)/2$ replaces λ_j because half of the relevant couplings equal $J_{2,i}$ and half of them equal $J_{2,j}$. This example shows that a singularity of the forms (14) and (15) occurs near a dense set of temperatures in the interval $T_{c,1} \leq T \leq T_G$. It implies that the nonanalytic behavior found by Griffiths for small fields is also reflected in the temperature behavior at zero field.

A different, one-sided form, $F_{\text{sing}} \sim t^{1/t}$, of Griffiths singularities in MW models was recently reported by Shankar and Murthy.[19] They derive the Griffiths singularity at the critical temperature $T_{c,1}$ connected to the weakest bonds. The form of the singularity is assumed to follow from Derrida-Hilhorst[20] singularities in random field Ising chains. However, the assumption (3.10) of Ref. 19(b) misses, e.g., an oscillating factor following from the more complicated exact result (4.23) of Nieuwenhuizen and Luck.[21] This questions the conclusion drawn in Ref. 19.

In case of *uncorrelated* random horizontal couplings, still taking a finite number of values $J_{2,j}$, the Lifshitz band tail (11) is determined by the minimum of the action

$$A(E) = \sum_{n,m} \left\{ -(1+E)\varphi(n,m)^2 + \sum_{m'} \left[(-\zeta)^{|m-m'|} \left(\frac{1+\zeta^2}{1-\zeta^2} - |m-m'| \right) \varphi(n,m)\varphi(n,m') \right] - \ln\left[\sum_{j=1}^{k} c_j \exp\left[-\left\{ z_j \varphi(n,m) - \sum_{m'} (-\zeta)^{|m-m'|} \varphi(n-1,m') \right\}^2 \right] \right] \right\}. \tag{16}$$

For T close to $T_{c,i}$ the term with $j=i$ will again dominate the logarithm in the region of large φ. Here the continuum equation becomes $-\xi^2\varphi_{xx}-\eta^2\varphi_{yy} =(E-t^2)\varphi$, with parameters as in (13). The solution with lowest energy is proportional to the Bessel function J_0. The leading contribution to A therefore gives

$$\rho(t^2+\epsilon)\sim\exp(-\lambda_i v_2/\epsilon)\,, \tag{17}$$

where $v_2=\xi\eta\pi\mu_2^2$ is the normalized volume of the relevant elliptic region of pure couplings, with $\mu_2 =2.40483$ being the first zero of the Bessel function J_0. Equations (10) and (17) prove the assertion (3) for the case of uncorrelated disorder. As in the MW case, this singularity occurs symmetrically near all critical temperatures $T_{c,j}$ of related pure systems.

Also for uncorrelated disorder one will find a similar Griffiths singularity near the critical point of any pure system with a larger unit cell, e.g., for a checkerboard arrangement of couplings $J_{2,i}$ and $J_{2,j}$. Consider, for instance, the above discussed case with $J_{2,i}$ in even rows and $J_{2,j}$ in the odd rows, but now for uncorrelated disorder. Going through the analysis we find that (17) remains valid with, however, ξ and $t\sim T-T_{c,ij}$ defined as in the corresponding problem with correlated disorder, discussed above, and with $(\lambda_i+\lambda_j)/2$ again replacing λ_i.

In conclusion, without giving a rigorous proof of their existence, we have derived the explicit, asymptotically exact form of Griffiths singularities in the free energy of two-dimensional Ising models in zero field, with correlated or uncorrelated random horizontal couplings. They are logarithmic transforms of Lifshitz tails in the density of states of a certain random matrix. Its form is connected to a one-component field in a random potential, which seems simpler than current approaches involving two-component spinors. δ functions in the density of couplings lead to symmetric Griffiths singularities, which have a universal form. They occur near temperatures connected to criticality of any periodic arrangements of random couplings. This leads to a dense set of singularities in the Griffiths phase $T_{c,1}\leq T\leq T_G$. As a result, the free energy is a nonanalytic function of temperature in this segment. The singularities overlap and the symmetric form only applies to the leading singular behavior, also at T_G. Griffiths singularities are found above as well as below the critical temperature T_c, where ferromagnetism sets in. In the study of analytic properties of the magnetization,[1] however, Griffiths singularities below T_c are masked by the singularity of the order parameter at zero field, already present in the pure system. Nevertheless, they give rise to critical slowing down in dynamics both for $T_c<T<T_G$ and for $T_{c,1}<T<T_c$.

For having a better description of the Griffiths singularities in case of uncorrelated disorder, it would be desirable to know the form of Lifshitz tails outside the asymptotic region in two-dimensional disordered systems. For a specific case in three dimensions this was discussed in Ref. 8. Another interesting equation is what happens for continuous distributions of disorder. For example, for a uniform distribution one expects the same location of singularities as for a binary distribution where only the weakest and the strongest of the couplings are retained; the form [Eqs. (10), (14), and (17)] of the singularity, however, is expected to involve $\ln(1/\epsilon)$ rather than λ_i.[9,10]

We hope that our results can be used for obtaining a better understanding of the long-time dynamics, and that our simplified form for the free energy will simulate theoretical and numerical work.

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