Farey Tree and Devil's Staircase of a Modulated External-Cavity Semiconductor Laser

Dieter Baums, Wolfgang Elsässer, and Ernst O. Göbel
Fachbereich Physik, Philipps-Universität Marburg, Renthof 5, D-3550 Marburg, Germany
(Received 29 March 1989)

We report frequency locking at Farey fractions of an electrically modulated semiconductor laser within an external cavity. The winding numbers as a function of the ratio of the modulation frequency to the inverse resonator round-trip time show the heirarchy of a Farey tree and the structure of a devil's staircase. The dimension of the set complementary to the stairs is determined to be 0.89. This demonstrates that the external-cavity semiconductor laser exhibits the universal properties characteristic for nonlinear systems driven by two competing frequencies.

PACS numbers: 42.50.Tj, 05.45.+b, 42.55.Px, 42.60.Fc

Nonlinear dynamics and chaos have been studied in many systems belonging to very different areas such as biology, chemistry, and physics. One of the most fascinating aspects is the search for universal laws related to the transition to chaos. The investigation of nonlinear dynamics and chaos in lasers is of particular interest[1] because, on the one hand, lasers represent active systems providing internal gain, and, on the other hand, well established means for the characterization and investigation of the laser output are available.

Some of the lasers described by the full set of three ordinary nonlinear differential equations of the Lorenz system for the electric field, medium polarization, and population inversion are expected to show the Lorenz-Haken–type instability which is characterized by a bifurcation route.[2] Semiconductor lasers are not considered to belong to this class of lasers because the lasing transition can be described by only two independent variables, namely the electric field and population inversion, while the third one, the material polarization, can be adiabatically eliminated. However, external modulation of the population inversion, which can be very easily accomplished in semiconductor lasers via the electrical driving current, or feedback of laser light, may provide an additional degree of freedom and chaotic behavior should be possible. In fact, the occurrence of chaos has been reported for external-cavity semiconductor lasers.[3,4] Introducing this extra degree of freedom adds a new characteristic frequency to the nonlinear system which is known to show frequency competition. The theoretical model describing nonlinear systems with two competing frequencies[5] predicts frequency locking and quasiperiodicity following the hierarchy of the Farey tree and the structure of the devil's staircase.[6] Frequency-locking phenomena have been demonstrated for a modulated self-pulsing laser diode[7] as well as for the modulated semiconductor in an external resonator.[8]

In this Letter we report for the first time frequency locking at pure Farey fractions $p/q<1$ (p and q are natural numbers) in the modulated external-cavity semiconductor laser. We find the features predicted mathematically by the one-dimensional discrete circle map. In particular, frequency-locked states occur in the sequence they appear in the Farey tree and within a frequency interval given by the width of the corresponding step in the devil's staircase. We further demonstrate that the time-averaged light output of the laser is a very sensitive indication of frequency-locked states. The modulated external-cavity semiconductor laser thus may serve as a simple model system to study the universal behavior of a nonlinear system characterized by two competing frequencies.

A GaAs/GaAlAs semiconductor laser (Hitachi HLP1400) which is antireflection coated on one side with a residual reflectivity lower than 10^{-4} is used in the experiments. The linear resonator consists of the remaining uncoated facet of the laser diode with a natural reflectivity of 32%, a high-reflectivity (98%) end mirror, an interference filter for wavelength selection, and a microscope objective for collimating the beam. The length of the resonator $L=1.5$ m is chosen to give an inverse round-trip time of $1/T=f_{\rm res}=c/2L=100$ MHz. The cw lasing threshold of the external-resonator setup is comparable to the original threshold of the uncoated laser. The limit power versus current characteristic does not show any irregularities and the laser is not self-pulsing. A tunable rf generator followed by a power amplifier is used to generate the sinusoidal modulation current which is superimposed on the dc current in a rf bias tee.

The light power output of the external-cavity laser is measured and time integrated with a slow p-i-n detector. The laser emission is temporally analyzed using a fast avalanche-photodiode (APD) detector with 10-GHz bandwidth followed by a sampling oscilloscope, and in the frequency domain by using the same APD together with an electrical spectrum analyzer with 2-GHz bandwidth. The present experimental setup corresponds to the standard configuration for synchronous active mode locking of the semiconductor laser.[9] In fact, clean synchronous mode-locked pulses are generated if the frequency $f_{\rm mod}$ of the current modulation is equal to an in-

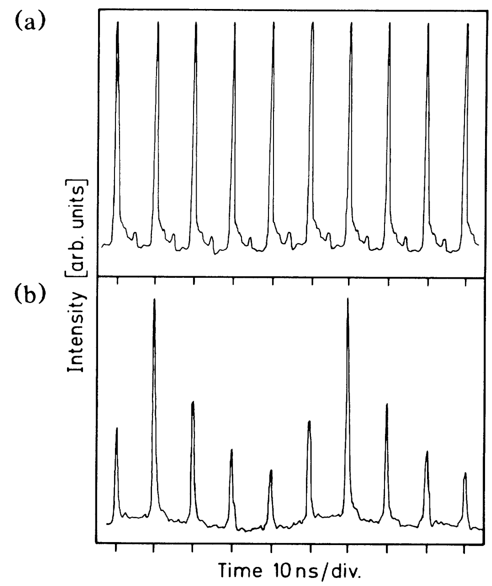


FIG. 1. Output pulse trains for (a) $f_{\mathrm{mod}}=f_{\mathrm{res}}$ (fundamentally mode locked) and (b) $f_{\mathrm{mod}}=\frac{2}{5}f_{\mathrm{res}}$ (frequency locked).

teger multiple of the inverse external-cavity round-trip time f_{res}. We then observe a regular and stable pulse train of pulses with a typical width of 46 ps (Ref. 10) and a repetition rate corresponding to f_{mod}.

In addition to these commonly studied mode-locked states, we have concentrated on the investigation of the laser output under modulation conditions where the frequency ratio $f_{\mathrm{mod}}/f_{\mathrm{res}}$ is a rational number, corresponding to frequency-locked states.

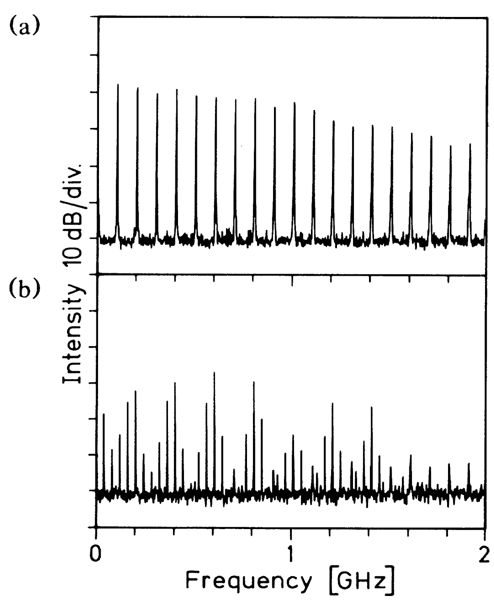


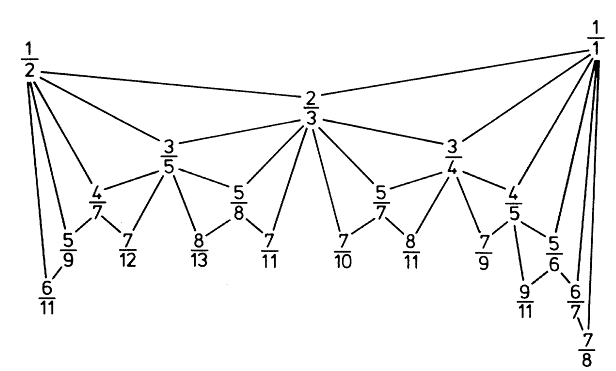
FIG. 2. rf spectra of the laser emission for (a) $f_{\mathrm{mod}}=f_{\mathrm{res}}$ (fundamentally mode locked) and (b) $f_{\mathrm{mod}}=\frac{2}{5}f_{\mathrm{res}}$ (frequency locked).

We have been able to detect pulse patterns belonging to $f_{\mathrm{mod}}/f_{\mathrm{res}}$ equal to fractions $n+p/q$ with $n=0,1$ and $p=1$ to q for q up to 6.[11] In Fig. 1 we compare the time-resolved laser output for the fundamental mode locking $(n+p/q=1)$ [Fig. 1(a)] with the frequency-locked state corresponding to the Farey fraction with $n=0$, $p=2$, $q=5$ [Fig. 1(b)]. The periodicity corresponds to the resonator round-trip time T in the upper case, whereas in the lower case the periodicity time interval is $5T$. To our knowledge this is the first report of frequency locking at *pure* Farey fractions $(n=0)$ in this system.

A more sensitive method for detecting a $(nq+p)/q$ frequency-locked state is to look for a coincidence of the $(nq+p)$th harmonic of the inverse resonator round-trip time and the qth harmonic of the modulation frequency. This coincidence is fulfilled whenever the qth harmonic of the modulation frequency falls into a certain interval around the $(nq+p)$th harmonic of the inverse resonator round-trip time. The width of the interval is of the order of a few megahertz and is defined by the frequency width of the amplitude and phase noise bands[12] and the appearance of sidebands as observed with the detuned mode-locked semiconductor laser.[13] The spectra taken from the rf spectrum analyzer simultaneously with the traces of Fig. 1 are shown in Figs. 2(a) and 2(b) for the fundamental mode-locked and the $\frac{2}{5}$ frequency-locked states, respectively. The coinciding frequency $(nq+p)f_{\mathrm{res}}=qf_{\mathrm{mod}}$ and its harmonics are the most intense lines in the spectra. However, all the harmonics of f_{mod} and f_{res} are excited. Table I lists all the Farey fractions identified through the rf spectra in the range of $f_{\mathrm{res}}/2\le f_{\mathrm{mod}}\le f_{\mathrm{res}}$.

The width of the interval for the modulation frequency which gives a p/q frequency-locked state depends on the level where p/q appears in the hierachy of the Farey tree. Plotting the Farey fractions of the frequency-

TABLE I. Part of the Farey tree containing the observed Farey fractions.



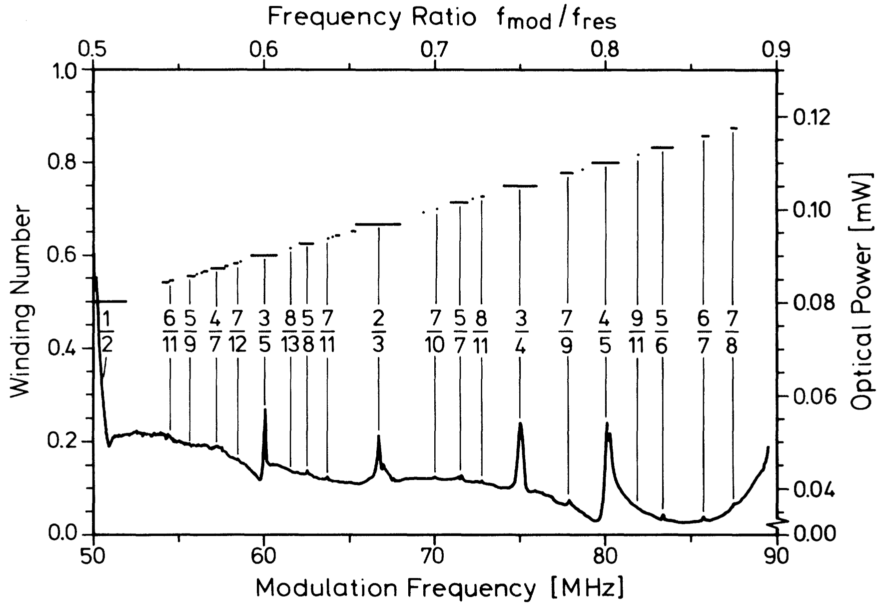


FIG. 3. Measured winding number (upper curve belonging to left vertical axis) and emitted optical power (lower curve belonging to right vertical axis) as a function of the modulation frequency f_{mod}. The upper horizontal axis shows the corresponding frequency ratio f_{mod}/f_{res}.

locked states as the winding numbers against the modulation frequency yields the upper curve in Fig. 3, revealing a devil's staircase. The winding numbers appear from left to right in the order predicted by the Farey tree. The gaps between the stairs refer to quasiperiodicity. We have employed the equation[14]

$$\sum_i (S_i/S)^D = 1 ,$$

for computation of the dimension D of the set of gaps. S in this equation refers to the distance between two stairs p/q and p'/q' while the S_i correspond to the lengths of the gaps between p/q and p'/q'. For $p/q = 1/2$ and $p'/q' = 1/1$ the dimension is calculated to be $D = 0.890 \pm 0.001$ which is very close to the value of 0.87 expected for the complete devil's staircase.[7] In addition, we show in Fig. 3 the simultaneously recorded optical output power for the same modulation frequencies. A comparison with the upper curve reveals that some of the stairs of the staircase coincide with a local maximum of the emitted optical power, thus providing a new and quick measurement of frequency-locked states for such a system with characteristic frequencies of the order of gigahertz. We have subsequently been able to detect *all* frequency-locked states expected according to the Farey tree up to an order of $q = 11$ between $\frac{1}{2}$ and $\frac{7}{8}$ on the basis of the spectral and time-integrated analysis of the laser output. This procedure actually is by far more sensitive for a distinction between frequency-locked and quasiperiodic states than merely looking for a pulsed output.

In conclusion, we have demonstrated that the electrically modulated semiconductor laser within an external cavity is an active system showing frequency locking at pure Farey fractions up to high orders. The higher orders are easily accessible through measurement of the emitted optical power or by an analysis of the rf frequency spectrum of the laser emission. The dimension of 0.89 determined from the measured devil's staircase indicates the universal nature of this nonlinear system. The modulated external-cavity semiconductor laser thus may serve as a particularly simple model system for the investigation of universal nonlinear properties.

We would like to thank J. O'Gorman for stimulating discussions. The fabrication of the antireflection coatings on the lasers by M. Serényi and G. Zimmermann is gratefully acknowledged. This work was supported by the Deutsche Forschungsgemeinschaft within the Sonderforschungsbereich SFB 185 (Nonlinear Dynamics) and by the European Council under Contract No. ST 2J-0276-C (CD).

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lower level of the tree by adding the numerators and the denominators separately (e.g., $\frac{1}{3}+\frac{1}{2}=\frac{2}{5}$). A devil's staircase is a curve with an infinite number of steps showing self-similarity. For a detailed review see, e.g., P. Bak, Phys. Today **39**, No. 12, 38 (1986).

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