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Coherent Structures in Multidimensions

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We solve an initial-boundary value problem for the Davey-Stewartson equation, a multidimensional analog of the nonlinear Schrödinger equation. It is shown that for large time, an arbitrary initial disturbance will, in general, decompose into a number of two-dimensional coherent structures. These structures exhibit interesting novel features not found in one-dimensional solitons.

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We consider an initial-boundary value problem for the Davey-Stewartson (DS)[1] system of equations

$$iq_t+\tfrac{1}{2}(q_{xx}+q_{yy})-(\varphi_x+|q|^2)q=0\,,$$
$$\varphi_{xx}-\varphi_{yy}+2|q|^2_x=0\,. \qquad (1)$$

The above system is the shallow-water limit of the Benney-Roskes equation,[2] where q is the amplitude of a surface wave packet while φ characterizes the mean motion generated by this surface wave. (One assumes a small-amplitude, nearly monochromatic, nearly one-dimensional wave train with dominant surface tension.[3]) Equation (1) provides a two-dimensional generalization of the celebrated nonlinear Schrödinger equation. Furthermore, it arises generically in both physics and mathematics. Indeed, it has been shown that a very large class of nonlinear dispersive equations in 2+1 (two spatial and one temporal) dimensions reduce to the DS equation in appropriate but generic asymptotic considerations.[4] Physical applications include water waves, plasma physics, and nonlinear optics.

We find it convenient to introduce characteristic coordinates $\xi=x+y$, $\eta=x-y$, and $U_1\equiv-\varphi_\eta-\frac{1}{2}|q|^2$, $U_2\equiv-\varphi_\xi-\frac{1}{2}|q|^2$; then the second equation in (1) can be integrated and Eqs. (1) reduce to

$$iq_t+(q_{\xi\xi}+q_{\eta\eta})+(U_1+U_2)q=0\,,$$
$$U_1=\tfrac{1}{2}\int_{-\infty}^{\xi}d\xi'\,|q|^2_\eta+u_1\,, \qquad (2)$$
$$U_2=\tfrac{1}{2}\int_{-\infty}^{\eta}d\eta'\,|q|^2_\xi+u_2\,,$$

where $u_1(\eta,t)\equiv U_1(-\infty,\eta,t)$, $u_2(\xi,t)\equiv U_2(\xi,-\infty,t)$. In this note we use the inverse scattering transform (IST) method to solve Eqs. (2), where $q(\xi,\eta,0)$, $u_1(\eta,t)$, and $u_2(\xi,t)$ are given, are decaying for large values of ξ, η, and are bounded for large values of t.

The first multidimensional equation solved in 2+1 dimensions was the three-wave interactions.[5] However, the relevant approach was based on the nondispersive nature of this system. The IST method was extended to dispersive equations in 2+1 dimensions in Refs. 6–8. A disappointing feature of all dispersive multidimensional equations studied so far has been the lack of two-dimensional exponentially decaying solitons. Taking into consideration the important role played by solitons in the applications of one-dimensional solvable systems, the above drawback has hampered considerably, in our opinion, the applicability of the exactly solvable multidimensional equations.

The special case of $u_1=u_2=0$ was solved in Ref. 8; in this case,[9] which is typical of what has occurred so far with multidimensional problems, arbitrary initial data disperse away as $t\to\infty$. We were motivated to reexamine Eqs. (2) because of the remarkable discovery of Boiti *et al.*[10] that the DS equations as well as a modified Korteweg–de Vries equation in 2+1 dimensions admit exponentially localized solitons. The above authors used Backlund transformations to obtain these solitons and investigated the dynamics of a certain two-soliton solution obtained using a nonlinear superposition formula, showing that the only effect of the interaction of the corresponding solitons is a two-dimensional phase shift. Our new investigation of Eqs. (2) shows that if u_1,u_2 are nonzero then energy from the mean flow can be transferred to the surface waves where it can create focusing effects. The mathematical manifestation of this phenomenon is the fact that DS equations admit localized solutions which decay exponentially in both ξ and η. Furthermore, any initial disturbance $q(\xi,\eta,0)$ decomposes to such solutions as $t\to\infty$. Indeed, (i) for arbitrary time-independent boundary conditions, any arbitrary initial disturbance will decompose into a number of *two-dimensional breathers* as $t\to\infty$. (ii) Similarly, for arbitrary time-dependent boundary conditions, any arbitrary initial disturbance will decompose into a number of *two-dimensional traveling localized structures* as $t\to\infty$. A simple such solution of (2) is[10]

$$q=\frac{4\rho(\mu_R\lambda_R)^{1/2}\exp\{-\mu_R\hat{\xi}-\lambda_R\hat{\eta}+i[\mu_I(\hat{\xi}+\bar{\xi})+\lambda_I(\hat{\eta}+\bar{\eta})+(|\mu|^2+|\lambda|^2)t+\arg(c\bar{c})]\}}{[1+\exp(-2\mu_R\hat{\xi})][1+\exp(-2\lambda_R\hat{\eta})]+|\rho|^2}, \tag{3}$$

where $\mu_R,\lambda_R\in R^+$, $\bar{\xi},\bar{\eta}\in R$, $\mu,\lambda,\rho,c,\bar{c}\in C$, $\hat{\xi}=\xi-2\mu_It-\bar{\xi}$, and $\hat{\eta}=\eta-2\lambda_It-\bar{\eta}$. A simple breather solution corresponds to $\mu_I=\lambda_I=0$.

Equation (1) is the reduction $r=-q^*$ of a more general system.[8] This system is associated with the Lax equation $(\partial_x+J\partial_y)\Psi+Q\Psi=0$, where $J=\mathrm{diag}(1,-1)$ and Q is an off-diagonal matrix containing the potentials q,r. In what follows we first recall the solution of an inverse problem associated with this equation. Namely, we reconstruct Q in terms of appropriate scattering data $S(k,l),T(k,l)$. We also consider the case of degenerate S,T, since this case is important for the discussion of solitons. We then show that if q,r satisfy the more general system mentioned above, the evolution of S,T depends crucially on u_1,u_2. Finally, we use certain completeness arguments to solve the equations satisfied by S,T in terms of u_1,u_2 and initial data.

(1) Using characteristic coordinates and letting $\Psi=\exp[ik(Jx-y)]M$, the Lax equation for $M(\xi,\eta,k)$ is solved by the following linear integral equations:

$$\begin{aligned}
M_{11}^{+}&=1-\tfrac{1}{2}\int_{-\infty}^{\xi}d\xi' qM_{21}^{+},\\
M_{12}^{+}&=-\tfrac{1}{2}\int_{-\infty}^{\xi}d\xi' qM_{22}^{+}\exp[ik(\xi-\xi')],\\
M_{21}^{+}&=\tfrac{1}{2}\int_{\eta}^{\infty}d\eta' rM_{11}^{+}\exp[ik(\eta'-\eta)],\\
M_{22}^{+}&=1-\tfrac{1}{2}\int_{-\infty}^{\eta}d\eta' rM_{12}^{+}.
\end{aligned} \tag{4}$$

Equations (4) are Volterra integral equations; thus for q and r in an appropriate space, M^+ is analytic in the upper half k-complex plane. Similarly, if M^- satisfies equations similar to those of (4), with the integrals in M_{21}^+,M_{12}^+ replaced by $\int_{-\infty}^{\eta}$ and $\int_{\xi}^{\infty}$, respectively, it follows that M^- is analytic in the lower half k-complex plane. Let $\mu^\pm$, $\hat{\mu}^\pm$ denote the vectors $\mu^\pm\equiv(M_{11}^\pm, M_{21}^\pm)^T$, $\hat{\mu}^\pm\equiv(M_{21}^\pm,M_{22}^\pm)^T$. Then M^+,M^- are related via the scattering equations

$$\begin{aligned}
\mu^+(k)-\mu^-(k)&=\int_R dl\, T(k,l)e^{-il\xi-ik\eta}\hat{\mu}(l),\\
\hat{\mu}^-(k)-\hat{\mu}^+(k)&=\int_R dl\, S(k,l)e^{il\eta+ik\xi}\mu^-(l),
\end{aligned} \tag{5}$$

where the scattering data T,S are given by

$$\begin{aligned}
T(k,l)&\equiv\frac{1}{4\pi}\int_{R^2}d\xi\, d\eta\, rM_{11}^{+}e^{ik\eta+il\xi},\\
S(k,l)&\equiv\frac{1}{4\pi}\int_{R^2}d\xi\, d\eta\, qM_{22}^{-}e^{-ik\xi-il\eta}.
\end{aligned} \tag{6}$$

Equations (5) define a nonlocal Riemann-Hilbert problem, the solution of which yields M^+,M^- in terms of T,S. Then q,r follow from

$$q=\frac{1}{\pi}\int_{R^2}dk\, dl\, S(k,l)M_{11}^{-}\exp(il\eta+ik\xi).$$

In the case that T,S are degenerate, the above formalism yields a closed-form solution.[11] It is now more convenient to work with the Fourier transform of S,T. Let $T(k,l)=\sum_{j=1}^{N_1}T_j(k)\tilde{T}_j(l)$, $S(k,l)=\sum_{j=1}^{N_2}S_j(k)\tilde{S}_j(l)$, and let $\tau,\tilde{\tau},\sigma,\tilde{\sigma}$ denote the Fourier transforms of $T,\tilde{T},S,\tilde{S}$. Then the Riemann-Hilbert problem (5) can be solved in closed form and q is given by $q=\pi^{-1}\sum_{j=1}^{N_1}\sigma_j(M_j)_1$, where $(M_j)_1$ is the first component of the vector M_j, which satisfies the following linear algebraic system:

$$\begin{aligned}
M_v&=\begin{pmatrix}1\\0\end{pmatrix}\tilde{\sigma}_v(\eta)-\frac{1}{2\pi}\sum_{j=1}^{N_1}\int_{-\infty}^{\eta}d\rho\, \tau_j(\rho)\tilde{\sigma}_v(\rho)\hat{M}_j,\\
\hat{M}_v&=\begin{pmatrix}0\\1\end{pmatrix}\sigma_v(\xi)-\frac{1}{2\pi}\sum_{j=1}^{N_2}\int_{-\infty}^{\xi}d\rho\, \tilde{\tau}_v(\rho)\sigma_j(\rho)M_j.
\end{aligned} \tag{7}$$

(2) The associated t part of the Lax pair is given by

$$\Psi_t=iJ(\partial_\xi-\partial_\eta)^2\Psi-iQ(\partial_\xi-\partial_\eta)\Psi+A\Psi, \tag{8}$$

where Ψ is the vector $(\Psi_1,\Psi_2)^T$, $A_{11}=iU_1$, $A_{12}=-iq_\eta$, $A_{21}=ir_\xi$, and $A_{22}=-iU_2$. However, in the presence of nontrivial boundary conditions Eq. (8) must be modified. The situation is conceptually analogous to the case of the nonlinear Schrödinger for $x\in[0,\infty)$; a nontrivial boundary condition at $x=0$ implies a modification of the associated t part of the Lax pair.[12] The precise modification depends on the form of the integral equation satisfied by Ψ. For concreteness let $\Psi_1\equiv e^{ik\xi}M_{12}^{-}$, $\Psi_2\equiv e^{-ik\xi}M_{22}^{-}$. It can be shown[13] that the time evolution of Ψ is given by an equation similar to (8) where the vector v is added on the right-hand side of (8). The vector v satisfies an equation similar to the one satisfied by Ψ where the forcing is replaced by $e^{ik\xi}[-ik^2+iu_2]$. Thus

$$v=-ik^2\Psi+\int_{-\infty}^{\infty}dl\,\gamma(k-l)\Psi(l)\,,\qquad \gamma(k)\equiv\frac{i}{2\pi}\int_{-\infty}^{\infty}d\xi\,e^{-ik\xi}u_2(\xi,t)\,. \tag{9}$$

Having obtained the correct t part of the Lax pair, it is now straightforward to obtain the time evolution of the scattering data. Indeed, if $\sigma(k,\eta)\equiv\int_R dl\,e^{il\eta}S(k,l)$, then $\sigma(k,\eta)=\lim_{\xi\to-\infty}\Psi_1$; hence σ satisfies

$$\sigma_t=iu_1\sigma+i\sigma_{\eta\eta}-ik^2\sigma+\int_R dl\,\gamma(k-l)\sigma(l,\eta)\,.$$

Thus if $\hat{S}(\xi,\eta)$ denotes the Fourier transform of $S(k,l)$, σ solves

$$i\hat{S}_t+\hat{S}_{\xi\xi}+\hat{S}_{\eta\eta}+(u_2+u_1)\hat{S}=0\,,\qquad \hat{S}(\xi,\eta)\equiv\int_{R^2}dk\,dl\,e^{ik\xi+il\eta}S(k,l)\,. \tag{10}$$

Similarly, for $\hat{T}$ where i is replaced by $-i$. Equations (5) and (10) provide in principle the solution of the initial-boundary value problem of DS. Given $q(\xi,\eta,0)$, $u_1(\eta,t)$, and $u_2(\xi,t)$, Eqs. (10) yield $S(k,l,t)$; similarly one obtains $T(k,l,t)$. Then the solution of the Riemann-Hilbert problem (5) yields M^- and then $q(\xi,\eta,t)$ follows.

(3) We now concentrate on the case $r=-q^*$, and we first consider the solution of Eq. (10) in the case that u_1,u_2 are t independent. We note that Eq. (10) is the linear limit of (2). Using separation of variables it follows that the solution of Eq. (10) is intimately related to the analysis of the stationary Schrödinger equation $\Psi_{xx}+[u(x)+k^2]\Psi=0$, where $u\in R$. We recall that this equation plays an important role in the integrability of the Koretweg-de Vries (KdV) equation, and it has been well studied; see, for example, Ref. 3: An arbitrary decaying function $u(x)$ gives rise to N discrete eigenvalues, $k_j=ip_j$, $p_j\in R^+$; furthermore, the discrete, $\varphi_j(x)$, and continuous eigenfunctions, $\varphi(x,k)$, form a complete orthonormal set. An arbitrary L_2 function $f(x)$ can be expanded in the form

$$f(x)=\sum_{n=1}^{N}\rho_n\varphi_n(x)+\int_R dk\,\rho(k)\varphi(x,k)\,,$$

$$\rho_n\equiv\int_R dx\,\varphi_n^*(x)f(x)\,,$$

$$\rho(k)\equiv\int_R dx\,\varphi^*(x,k)f(x)\,.$$

If the reflection coefficient of the potential u is zero, then the discrete eigenfunctions can be found in closed form,

$$\varphi_n+\sum_{j=1}^{N}\frac{c_nc_j}{p_n+p_j}e^{(p_n+p_j)x}\varphi_j=c_ne^{p_nx}\,,\qquad u=2\sum_{n=1}^{N}c_n[e^{p_nx}\varphi_n(x)]_x\,. \tag{11}$$

Using the above results about the stationary Schrödinger equation it follows that

$$\hat{S}(\xi,\eta,t)=\sum_{n=1,m=1}^{N,M}\rho_{nm}X_n(\xi)Y_m(\eta)e^{i(\mu_n^2+\lambda_m^2)t}+\int_{R^2}dk\,dl\,\rho(k,l)X(\xi,k)Y(\eta,l)e^{-i(k^2+l^2)t}$$
$$+\int_R dk\left[\sum_{n=1}^{N}\rho_n(k)e^{i(\mu_n^2-k^2)t}X_n(\xi)Y(\eta,k)+\sum_{m=1}^{M}\bar{\rho}_m(k)e^{i(\lambda_m^2-k^2)t}X(\xi,k)Y_m(\eta)\right]\,, \tag{12}$$

where $\{X(\xi,k),X_j(\xi),p_j=\mu_j;j=1,\ldots,N\}$ and $\{Y(\eta,k),Y_j(\eta),p_j=\lambda_j;\ j=1,\ldots,M\}$ are the orthonormal sets corresponding to $u_2(\xi)$ and $u_1(\eta)$, respectively. Furthermore, ρ_{nm}, $\rho_n(k)$, $\bar{\rho}_m(k)$, and $\rho(k,l)$ are obtained from the initial data via

$$\int_{R^2}d\xi\,d\eta\,\hat{S}(\xi,\eta,0)F^*(\xi,\eta,k,l)\,,$$

where for ρ_{nm}, $F=X_n(\xi)Y_m(\eta)$; for $\rho(k,l)$, $F=X(\xi,k)\times Y(n,l)$; for $\rho_n(k)$, $F=X_n(\xi)Y(\eta,k)$; and for $\bar{\rho}_m(k)$, $F=X(\xi,k)Y_m(\eta)$. Similarly for $\hat{T}(\xi,\eta,t)$, where t is replaced by $-t$.

The stationary-phase method implies the following asymptotic behavior in time:

$$\hat{S}(\xi,\eta,t)\sim\sum_{n=1,m=1}^{N,M}\rho_{mn}X_n(\xi)Y_m(\eta)\exp[i(\mu_n^2+\lambda_m^2)t],\quad \text{as } t\to\infty\,. \tag{13}$$

Thus as $t\to\infty$ the scattering data become degenerate and using (7) one obtains for q a localized solution which oscillates in time (breather). The asymptotic behavior of q can be calculated in closed form provided that X_n,Y_m can be found in closed form. For example, if u_1,u_2 are reflectionless, then X_n,Y_m satisfy the linear system (11), and q becomes what we call an (N,M) breather. From the above it follows that if at least one of the two boundaries does not give rise to bound states of the Schrödinger operator, then every initial condition $q(\xi,\eta,0)$ will disperse away. If bound states do exist, the asymptotic behavior is essentially determined by these bound states; the initial condition only fixes the constant ρ_{nm}.

If u_1,u_2 are time dependent, then separation of variables implies that the solution of Eq. (10) is now intimately related to the analysis of the time-dependent Schrödinger equation $i\Psi_t+\Psi_{xx}+[u(x,t)+k^2]\Psi=0$, $u\in R$. We recall that this equation plays an important role in the integrability of the Kadomtsev-Petviashvili

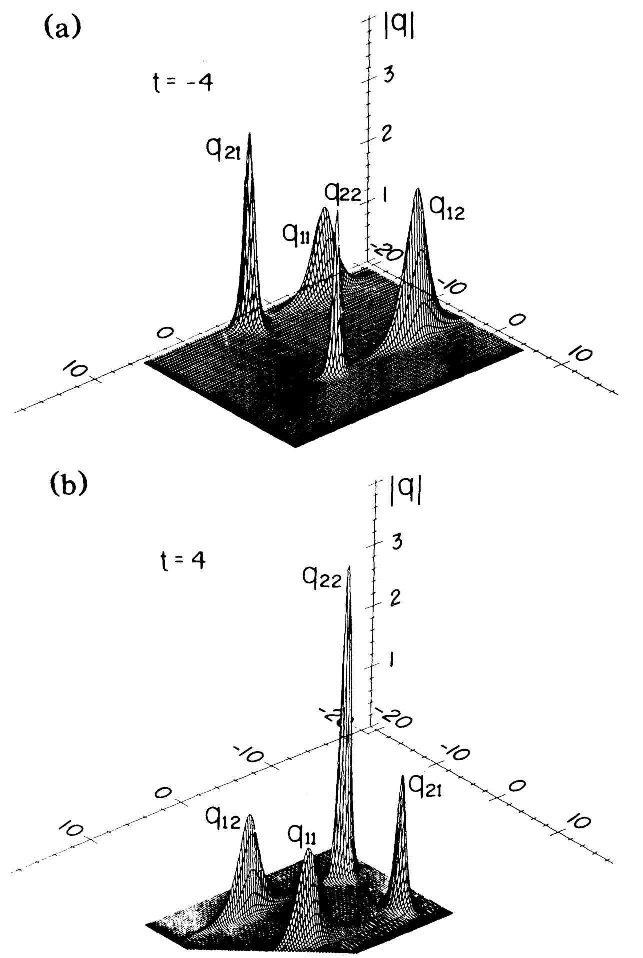

FIG. 1. The (2,2) traveling localized solution at (a) $t=-4$ and (b) $t=4$.

(KP) equation where t is replaced by y.[6,7] However, now we demand u to be decaying in x only, as opposed to the case of KP where u is decaying in both x and y. We expect that a completeness result is also valid for the above equation, so that Eq. (12) is appropriately generalized. Here we only note that in analogy with Eq. (11) we have

$$\varphi_n+\sum_{j=1}^{N}\frac{c_n c_j^*}{p_n+p_j^*}\exp\{-(p_n+p_j^*)[x-i(p_n-p_j^*)t]\}\varphi_j = c_n\exp[-p_n(x-ip_nt)]\,, \quad (14)$$

$$u=-2\partial_x\sum_{n=1}^{N}c_n^*\exp[-p_n^*(x+ip_n^*t)]\varphi_n\,.$$

Hence, as before, we expect

$$\hat{S}(\xi,\eta,t)\sim\sum_{n=1,m=1}^{NM}\rho_{nm}X_n(\xi,t)Y_m(\eta,t)\,,$$

Eq. (13) for large time, where if u_1,u_2 are "reflectionless," X_n,Y_m satisfy Eqs. (14) with $p_j=\mu_j,\lambda_j$, respectively. Then Eqs. (7) yield an (N,M) traveling localized solution; the (1,1) case is given by (3). We note that nontrivial contributions occur when $x+2p_lt\sim 1$.

It is quite remarkable that the stationary and nonstationary Schrödinger operators, which are crucial for the integrability of the KdV and KP equations, respectively, are also crucial for obtaining coherent solutions of the DS equation.

We note that Eq. (10) is the linear limit ($q\sim 0$) of Eq. (2), which is consistent with the fact that the inverse scattering transform reduced to the Fourier transform in the linear limit. Since the two-dimensional localized solutions are associated with the discrete spectrum of Eq. (10), it follows that they are nonlinear distortions of the bound states of the linearized equation. It turns out that in contrast to one-dimensional solitons these two-dimensional coherent solutions do not in general preserve their form upon interaction and exchange energy (only for a special choice of the spectral parameters these solutions preserve their form and the solutions of Ref. 10 are revoked).[14,15] Furthermore, these localized structures can be driven everywhere in the plane by choosing a suitable motion for the boundaries.[14] In Fig. 1 $|q|$ for the (2,2) traveling localized solution is plotted at $t=-4$ and 4 for specific values of the spectral parameters. We note that the form of each lump changes upon interaction (for details see Ref. 15).

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