Scattering Amplitudes in Hot Gauge Theories

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(Received 21 November 1988)

A consistent method for the perturbative calculation of scattering amplitudes in hot gauge theories is developed. It involves the resummation of a subset of thermal fluctuations, termed hard thermal loops, into effective propagators and vertices. The damping rate of a heavy fermion is computed as an example.

PACS numbers: 12.38.Mh, 12.38.Bx

The properties of QCD at a temperature T are of importance for understanding the collision of nuclei at ultrarelativistic energies.[1] Attention has focused recently on the properties near equilibrium of "hot" QCD, well into its chirally symmetric, deconfined phase. Particularly confusing is the infrared limit in perturbation theory: On its mass shell both the sign and the magnitude of the gluon damping rate appear to be gauge dependent.[2]

In this Letter I argue that whenever a quantity is calculated perturbatively in a hot gauge theory, sooner or later an infinite subset of diagrams, nominally of higher order in the loop expansion, contribute to the same order in the coupling constant g. These higher-loop diagrams are isolated and resummed into an effective expansion which includes all effects to leading order in g.

I start by distinguishing between hard and soft momenta. Let an external momentum for a diagram be P^μ, $P^\mu=(p_0,\bar{p})$, $p=|\bar{p}|$. In imaginary time, p_0 is an integral multiple of πT; an amplitude in real time is obtained by analytically continuing each p_0 to $p_0=-i\omega$, with ω a real, continuous variable. A momentum is defined as "soft" if ω and p are of order gT; a momentum is "hard" if either is of order T. The only soft lines in imaginary time are bosons with zero energy, but for amplitudes in real time, both boson and fermion lines can be soft.

When any external leg is hard, loop corrections to the tree amplitude are suppressed by at least one power of g. If every leg is soft, though, there are loop corrections which are in magnitude g^2T^2/P^2 times the corresponding tree diagram. These graphs, which I call "hard thermal loops," arise from one-loop integrals in which the loop momentum is hard. At hard P, hard thermal loops are simply part of the usual perturbative corrections, but for soft P they are as important as the tree diagram. Diagrammatically, hard thermal loops are produced by tadpole graphs and by the contribution of thermal pairs. In the latter, one member of the pair is absorbed from the thermal distribution—say with energy E_k—and the other emitted into it, with energy E_{p-k} (k represents the loop momentum). Thermal pairs produce energy denominators like $ip_0+E_k-E_{p-k}$: Even at hard k this denominator is soft, $\simeq ip_0+p\cos\theta$, and gT over it is of order 1. This is very different from zero temperature, where only emission and not absorption is allowed. At $T=0$, the energy denominators are like $ip_0-E_k-E_{p-k}$: This is hard if k is, and gT over it is of order g, not of order 1. Thermal pairs generate a discontinuity at spacelike momenta below the light cone, $p\gtrsim\omega\gtrsim -p$, which is Landau damping.[3] At high momentum, the distribution functions which accompany absorption are Boltzmann type and so exponentially damped; thus the integrals which produce thermal loops are ultraviolet finite.

For a (massless) scalar theory with quartic interactions equal to g^2, the hard thermal loop in the self-energy is a mass term, $m_s^2\sim g^2T^2$.[1] In hot QED, the hard thermal loop in the photon's self-energy was evaluated first by Silin;[4] in hot QCD, the hard thermal loops in the quark and gluon self-energies were computed initially by Klimov and Weldon.[5,6] For the scalar self-energy only the tadpole diagram contributes to the hard thermal loop, but in gauge theories both tadpolelike diagrams and thermal pairs contribute. Since the Landau damping of thermal pairs operates from $\omega=p$ down to $-p$, the fermion and gauge self-energies are nontrivial functions of momentum, as in Eq. (1). For instance, while the effective fermion and gauge propagators formed from these self-energies have mass shells which lie above the light cone by order gT, the effective mass shells do not have a relativistically invariant form.

For scalars, the only hard thermal loop is in the two-point function; e.g., the running coupling constant varies as the logarithm, and not as a power, of T. In gauge theories there are an infinite number of hard thermal loops: For QCD, between N gluons, and between a quark, antiquark, and $N-2$ gluons, at all $N\gtrsim 2$.[6,7] These are due to thermal pairs, and so like the self-energies they are nontrivial functions of the external momentum.

Consider now amplitudes which are g times smaller than those at tree level. These diagrams typically receive contributions from graphs in which at least some of the virtual lines have soft momenta. Since over soft momenta hard thermal loops are comparable to tree diagrams, a result is only complete after they have been resummed to all orders in the loop expansion in all possible ways. For soft legs it is necessary to use an effective propagator which includes the hard thermal loop in the self-energy. Likewise, if all of the momenta going into a vertex are soft, an effective vertex is needed, which includes both the tree vertex and the hard thermal loop.[6,7] In a scalar theory this is trivial: All that is needed is to replace the bare propagator, $1/P^2$, by $1/(P^2+m_s^2)$. For gauge theories, the effective propagators must incorporate the self-energies of Silin, Klimov, and Weldon exactly; moreover, many vertices are effective, and contain hard thermal loops.

With hard lines, or for a vertex in which any leg is hard, bare quantities suffice at leading order. Specifically, this includes everything inside a hard thermal loop. This guarantees that the process of resummation is consistent: While some loop effects are of order 1 at soft momenta, they arise entirely from hard loops. Corrections to a hard loop are small, down by order g.

These complications are special to hot theories, and do not occur in the cold limit. In hot theories the only scale that cuts off infrared divergences in loop diagrams is radiatively induced, of order gT, which is small relative to T. Presumably, hot fermion and gauge theories are so much more intricate than scalar theories because the former transform nontrivially under the Lorentz group, and in a thermal distribution the Lorentz symmetry is no longer manifest.

To illustrate the resummation of hard thermal loops, consider the effective gluon propagator. The contributions of the hard thermal loop to the gluon's longitudinal and transverse self-energies are[4-7]

$$\delta\Pi_l = -3m_g^2 Q_1\left(\frac{ip_0}{p}\right);$$
$$\delta\Pi_t = \frac{3}{5}m_g^2\left[Q_3\left(\frac{ip_0}{p}\right) - Q_1\left(\frac{ip_0}{p}\right) - \frac{5}{3}\right]. \tag{1}$$

$m_g \sim gT$ is the gluon "mass": $m_g^2=(N+N_f/2)(gT)^2/9$ for an SU(N) gauge group with N_f fermions in the fundamental representation. Q_1 and Q_3 are the Legendre functions of the second kind, and appear often in hard thermal loops.[7]

The hard thermal loops of Eq. (1), and for the quark self-energies, are the same in covariant and Coulomb gauges.[4-7] It is surprising to find that these hard thermal loops are gauge independent, for while the momenta must be soft, they need not be on mass shell. Of course, the effective propagator changes with gauge. In Coulomb gauge, $\partial_i A^i=0$, the effective gluon propagator is $^*\Delta_{00}={}^*\Delta_l$, $^*\Delta_{0i}=0$, and $^*\Delta_{ij}=(\delta^{ij}-p^ip^j/p^2)\,^*\Delta_t$, with $^*\Delta_l=1/(p^2-\delta\Pi_l)$ and $^*\Delta_t=1/(p_0^2+p^2-\delta\Pi_t)$.

Given the nontrivial momentum dependence of the effective propagators and vertices, an important question is how to compute with them in any practical sense. In ordinary perturbation theory the tedium of performing discrete sums over the Euclidean p_0 can be avoided by using a "noncovariant" approach.[3] This method can be generalized directly to the effective expansion. Propagators that depend upon p_0 and p are Fourier transformed into functions of the Euclidean time τ and p. This produces a spectral representation for the $^*\Delta$'s:[6,7]

$$^*\Delta_{l,t}(\tau,p) = T\sum_{\substack{j=-\infty \\ p_0=2\pi jT}}^{+\infty} e^{-ip_0\tau}\,^*\Delta_{l,t} = \int_0^\infty d\omega\,^*\rho_{l,t}(\omega,p)\{[1+n(\omega)]e^{-\omega\tau}+n(\omega)e^{+\omega\tau}\}\,. \tag{2}$$

$n(\omega)=1/[\exp(\omega/T)-1]$ is the Bose-Einstein distribution function, and $^*\rho_{l,t}$ is the spectral density,

$$^*\rho_{l,t}(\omega,p) = {}^*\rho_{l,t}^{\text{res}}(\omega_{l,t}(p),p)\,\delta[\omega-\omega_{l,t}(p)] + {}^*\rho_{l,t}^{\text{disc}}(\omega,p)\,\theta(p-\omega)\,,$$

with $\theta(x)=0,1$ for $x<0$, $x>0$. Each $^*\Delta$ has a single pole above the light cone, lying on the mass shell $\omega=\omega_{l,t}(p)$, with residue $^*\rho_{l,t}^{\text{res}}$. The discontinuity $^*\rho_{l,t}^{\text{disc}}$ below the light cone is Landau damping.

The usual transverse excitations of a gauge field are given by the transverse pole: It has a mass $m_g \sim gT$, with $\omega_t\,^*\rho_t^{\text{res}} \sim 1$ for all p. The longitudinal pole is a collective excitation: Although it also has a mass of order gT, its residue is only significant when $p \sim m_g$, and vanishes exponentially for $p \gg m_g$.

For the damping rates, what are most important are not the quasiparticle excitations in the $^*\Delta$'s, but the discontinuities. For nearly static gluons, $\omega \ll p$,

$$^*\rho_l^{\text{disc}}(\omega,p) \underset{\omega\to 0}{\simeq} \left(-\frac{1}{p^2}\right)\frac{3}{2}\frac{m_g^2\omega p}{(p^2+3m_g^2)^2}\,,$$
$$^*\rho_t^{\text{disc}}(\omega,p) \underset{\omega\to 0}{\simeq} \frac{3}{4}\frac{m_g^2\omega p}{[p^6+(3\pi m_g^2\omega/4)^2]}\,. \tag{3}$$

The effective propagators obey some general properties. The transverse spectral density is never negative, $^*\rho_t \geq 0$; in accordance with the equal-time commutation relations it satisfies a sum rule, $\int_0^\infty d\omega(2\omega)\,^*\rho_t(\omega,p)=1$. Sum rules for other moments of $^*\rho_t(\omega,p)$ with respect to ω, and for those of $^*\rho_l(\omega,p)$, can be derived by using the analytic properties of $^*\Delta_t$ and $^*\Delta_l$ in the complex p_0 plane.[7]

In contrast to the transverse spectral density, the longitudinal density is never positive, ${}^*\rho_l \le 0$; also, it is only smooth about zero momentum after an overall factor of $1/p^2$ is extracted, as in Eq. (3). The example of an Abelian gauge field in a background, conserved current[6] demonstrates that despite the $-1/p^2$ in ${}^*\rho_l$, its contribution to physical quantities is infrared finite and of positive weight. This is because in gauge-invariant quantities, ${}^*\Delta_t$ couples to the spacelike part of the current, and ${}^*\Delta_l$ to the timelike part.

Fermions are treated similarly. For massless fermions,[5,6] at positive energy ($\omega > 0$), the effective fermion propagator has not one, but *two* branches above the light cone. One branch is standard: It has a mass of order gT, a residue of order 1 for all p, and chirality equal to helicity. The second branch is a collective mode: Also above the light cone by order gT, its chirality is *minus* its helicity, with a residue that vanishes exponentially for $p \gg gT$.[6]

The two branches persist as a bare fermion mass is turned on. This is simple to compute if the mass m is soft, of order gT.[6] As m increases from zero, the standard branch becomes heavier and the collective mode lighter, but the residue of the collective mode decreases as well. By the time that m is of order T, the residue of the collective mode is $\le g^2$ at all momenta, and its effects are negligible.

At temperatures $T \simeq 100$–300 MeV in the quark-gluon plasma, the up and down quarks are essentially massless, with the strange-quark mass of order T. Thus the propagation of up and down quarks is strongly altered over soft momenta, and exhibits a collective mode with flipped chirality and helicity. For strange quarks, loop effects are suppressed by g^2, and the collective mode can be ignored.

Going beyond the self-energies to higher-point functions, the hard thermal loops can be isolated directly if the bare one-loop diagrams are evaluated by noncovariant means.[3] Braaten and I have explicitly computed the hard thermal loops for the three- and four-point functions of hot QCD;[7] in the end, they reduce to relatively simple forms like those for the self-energies, Eq. (1). We have also constructed generating functionals for the hard thermal loops of arbitrary N-point functions. Remarkably, all hard thermal loops are identical in covariant and Coulomb gauges for any value of the soft external momenta. Effective vertices are formed by adding the hard thermal loop to the bare vertex. Like the effective propagators, these can be incorporated into the noncovariant scheme; the spectral representation of the vertices, similar to Eq. (2), has no poles, only cuts from Landau damping.

In the calculations of the gluon damping rate in Ref. 2, the gluon is assumed to be at rest on its effective mass shell, $p=0$ and $\omega = m_g$. The damping rate is obtained from terms in the gluon self-energy, $\Pi^{\mu\nu}$, that are of order g times the hard thermal loop. In Ref. 2, bare propagators and vertices were used to compute $\Pi^{\mu\nu}$, which gives a "bare" damping rate. The bare damping rate receives corrections of order 1 from an infinite number of higher-loop diagrams: Arbitrary insertions of hard thermal loops within the bare one-loop diagram contribute to terms in $\Pi^{\mu\nu}$ that are still of order g.

The "effective" damping rate is computed from diagrams in the effective expansion which are topologically the same as in the bare expansion. At zero momentum every internal and external line is soft, so effective propagators and vertices are required throughout. The effective $\Pi^{\mu\nu}$ and the damping rate are again of order g, but because all hard thermal loops have been resummed, these results are *complete* to leading order in g. The price of resummation is that it is not easy to compute the effective damping rate for light fields at rest.[7]

A problem which typifies the physics of the damping rate for soft gluons, and yet is computationally much simpler, is the damping rate of a heavy quark, F. If F has a mass $M \gtrsim T$, it is automatically a hard field. Consider the effective one-loop diagram for the self-energy of F. The propagator for F, and its vertices, can all be taken as bare—the only effective quantity required is for the soft-gluon propagator, Eqs. (1)–(3). Part of the discontinuity of this effective graph is

$$\mathrm{Disc}\Sigma_F \simeq -\frac{g^2}{2}\pi C_F T \int_{\mathrm{soft}\,k} \frac{d^3k}{(2\pi)^3} \int_0^\infty \frac{d\omega}{\omega} \sum_{\substack{i=l,t\\ s=\pm}} {}^*\rho_i(\omega,k) G_i \delta(s\omega + E_p - E_{p-k})\,. \tag{4}$$

F is assumed to be on mass shell with positive energy: $P^\mu = (-iE_p, \bar{p})$, $E_p = (p^2+M^2)^{1/2}$; C_F is the Casimir for F, and $G_l \simeq \gamma^0 + 1$, $G_t \simeq 2(-\gamma^0+1)$. In Eq. (4) all terms that do not contribute to Eqs. (5) and (6) at order g^2 have been dropped. For example, Σ_F receives contributions from both soft and hard virtual gluons. By kinematics, though, a hard gluon cannot contribute to the damping rate on mass shell. I also took $n(\omega) \simeq T/\omega$, and neglected terms which are down by powers of k/M, etc.

When F is at rest, $p=0$, the only way to satisfy energy-momentum conservation is through the term with $s=+$ in Eq. (4), $\omega = E_k - M \simeq k^2/2M$. This forces the gluon to be nearly static, so the discontinuities of Eq. (3) apply. The transverse density ${}^*\rho_t$ produces a term in $\mathrm{Disc}\Sigma_F \simeq \gamma^0 - 1$. For nearly static gluons the transverse density is infrared singular, which produces powerlike infrared divergences in $\mathrm{Disc}\Sigma_F$. These contribute solely to wave-function renormalization, and so are not of physical consequence. The longitudinal density ${}^*\rho_l$ produces a finite contribution to $\mathrm{Disc}\Sigma_F \simeq \gamma^0 + 1$. This shifts the pole in the effective propagator, $-i\not{P} + M - \Sigma_F$, from its value at tree level, $\omega = M$, with the shift in the imaginary direction proportional to the damping rate. I define the quantity $\gamma \equiv -\mathrm{Im}\omega/\mathrm{Re}\omega$, evaluated at the pole in the effective propagator, as a dimensionless measure of the damping rate. At zero momentum,

$$\gamma \underset{p=0}{\simeq} +\frac{g^2 C_F}{8\pi}\frac{T}{M}. \quad (5)$$

Positivity of γ demonstrates that F is thermodynamically stable.

For ease of calculation, at nonzero momentum assume that F is nonrelativistic, $M \gg p \gg m_g$. The terms in Eq. (4) for $s=\pm$ both contribute, with $\omega = \pm(E_{p-k} - E_p) \simeq \mp pk\cos(\theta)/M$. Nearly static gluons, $\omega \ll k$, occur when the virtual gluon is emitted from F near 90°, $\theta \simeq \pi/2$. Integrating Eq. (4) with respect to ω and θ about this region, the contribution from the transverse density produces a logarithmic divergence in the integral over k. From Eq. (3) the logarithm is cut off above by m_g; I cut if off below at a mass scale m_{mag}. Then at nonzero momentum

$$\gamma \underset{p\neq 0}{\simeq} +\frac{g^2 C_F}{8\pi}\ln\left(\frac{m_g}{m_{\mathrm{mag}}}\right)\frac{pT}{M^2}. \quad (6)$$

Terms in γ from other regions of integration, and from the longitudinal density, are infrared finite, $\simeq g^2(pT/M^2)$. Equation (6) applies only for $p \gg m_g$, and crosses smoothly over to Eq. (5) when p is of order m_g.

For nearly static gluons ${}^*\rho_t(\omega,p)$ in Eq. (3) is infrared singular, and $p^2\,{}^*\rho_l(\omega,p)$ infrared regular, because to one-loop order static electric fields are screened, but static magnetic fields are not.[1,6] It is expected that nonperturbative effects screen static magnetic fields over distances of order $1/g^2T$.[1] A spectral sum rule for ${}^*\rho_t(\omega,p)/\omega$ can be used to show that $1/m_{\mathrm{mag}}$ is equal to the magnetic screening length.[7] Hence in Eq. (6), $m_{\mathrm{mag}} = c_{\mathrm{mag}} g^2 T$, with c_{mag} a gauge-invariant number of order 1, and $\ln(m_g/m_{\mathrm{mag}}) \simeq \ln(1/g)$.

The Coulomb gauge was used in Eq. (4), but Eqs. (5) and (6) are gauge invariant. To show this, form the two-point $\mathcal{T}$-matrix element from Σ_F by sandwiching it between two physical wave functions, $\mathcal{T}_F = \bar{\psi}\Sigma_F\psi$. The wave functions $\bar{\psi}$ and ψ are on mass shell, so terms in Σ_F which contribute to γ survive in $\mathcal{T}_F$, while those which contribute to wave-function renormalization drop out. It is a textbook exercise to use the bare Ward identities to show that in the effective one-loop graph for Σ_F, when the effective gluon propagator in Coulomb gauge is replaced by that in covariant gauge, Σ_F changes, but $\mathcal{T}_F$ does not. A parallel analysis can be used to demonstrate that at hard momentum ($p \gtrsim T$), on mass shell the effective damping rates for light quarks and gluons are gauge invariant; as in Eq. (6), $\gamma \simeq +g^2\ln(1/g)$.

Braaten and I have extended the proof of gauge invariance to the effective damping rates of light quarks and gluons, on mass shell at soft momentum.[7] Two-point $\mathcal{T}$-matrix elements are formed by sandwiching the self-energies between wave functions on their effective mass shell. Although the hard thermal loops are nontrivial functions of momentum, the effective two-, three-, and four-point functions which enter into the damping rates satisfy effective Ward identities identical in form to the bare Ward identities. The effective Ward identities are then used to establish that the two-point $\mathcal{T}$-matrix elements, and so the γ's, are equal in covariant and Coulomb gauges. The effective damping rates for soft, light fields are similar to those for a heavy fermion.[7] At rest, $\gamma \simeq +n(gT)g^2 \simeq +g$; at soft, nonzero momentum, $T \gg p \gg m_{\mathrm{mag}}$, $\gamma \simeq +g\ln(1/g)$.

I end by stressing that the effective expansion applies not just to damping rates, but to the computation of *arbitrary* amplitudes in hot gauge theories. Notable examples include the calculation of dilepton production,[7] transport coefficients, and the development of a consistent kinetic theory.[4]

I thank Dan Boyanovsky, Tetsuo Matsui, Larry McLerran, Art Weldon, and, especially, Eric Braaten for discussions. Fermilab is operated by Universities Research Association under contract with the United States Department of Energy.

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