Thermal Fluctuation and Melting of the Vortex Lattice in Oxide Superconductors

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The thermal fluctuation of the vortex positions in type-II superconductors with large Ginzburg-Landau parameter κ and weak pinning is drastically increased when calculated from the correct, nonlocal elasticity of the vortex lattice. As compared to the usual (local) elastic result, the mean-square shear strain and vortex displacements increase by a factor of $\approx (\bar{B}/B_{c1})^{1/2}/(1-\bar{B}/B_{c2})^{1/2}$ ($\bar{B}$ is the flux density, B_{c1} and B_{c2} are the lower and upper critical fields). The melting temperature of the vortex lattice is reduced by the same factor.

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The melting of the flux-line lattice (FLL) was discussed in detail first by Fisher[1] for thin superconducting films of thickness d much smaller than the bulk magnetic penetration depth λ. In such films the two-dimensional shear modulus of the FLL $c_{66}d$ is very small and thermal fluctuation of the flux-line (FL) positions may become large enough that the FLL melts at a temperature T_m considerably smaller than the superconducting transition temperature T_c. In principle, a bulk FLL should melt similarly when the applied field B_a is sufficiently close to the upper critical field $B_{c2}(T)$, where the shear modulus c_{66} vanishes as $(B_{c2}-B_a)^2$. In conventional superconductors, however, three-dimensional melting is expected only very close to B_{c2}, in a range of B_a or T which is practically not accessible since material inhomogeneities smear the transition as $B_{c2}(T)$ and, more importantly, since the always present pinning forces vanish only as $B_{c2}-B_a$ (proportional to the order parameter $|\psi|^2$).

Recently, Nelson[2] has pointed out that FLL melting should be observable in high-T_c superconductors since (a) high thermal energies k_BT are accessible and (b) the shear modulus of the FLL may become very small due to the large Ginzburg-Landau parameter $\kappa \approx 200$. From Lindemann's criterion one might expect melting to occur when the root-mean-square thermal average $\langle u^2\rangle^{1/2} = \langle u_x^2+u_y^2\rangle^{1/2}$ of the FL displacements $\mathbf{u}(\mathbf{R}_\perp,z)$ from their equilibrium positions $\mathbf{R}_\perp$ (z is the coordinate along the FL's) reaches $\approx a/10$ (a is the FL spacing). Nelson and Seung[3] find for the unpinned FLL

$$\langle u^2\rangle = \langle u^2\rangle_{\rm loc} = (n/4\pi c_{66}\hat{c}_{44})^{1/2} k_BT\,, \tag{1}$$

where $n=\bar{B}/\phi_0$ is the FL density and $\hat{c}_{44}=\bar{B}B_a/\mu_0$ is the modulus for uniform tilt of the FLL. Here I shall not consider the influence on $\langle u^2\rangle$ of the pronounced anisotropy of the high-T_c superconductors which in Ref. 3 is estimated to be huge. I only point out that in nearly the entire field range $B_a \gtrsim 2B_{c1}$ one has $\bar{B}\approx B_a$. Therefore, the "local elastic" tilt modulus $\hat{c}_{44}\approx \bar{B}^2/\mu_0$ equals twice the magnetic field energy which is *isotropic*. However, for nonuniform strains with finite wave vectors $\mathbf{k}=(\mathbf{k}_\perp,k_z)$, $\mathbf{u}(\mathbf{R}_\perp,z)=\mathbf{u}(\mathbf{k})\exp(i\mathbf{k}_\perp\cdot\mathbf{R}_\perp+ik_zz)$, the nonlocal elastic tilt modulus, as well as the uniaxial compressional modulus $c_{11}(k)$ and c_{66} in general, will be *anisotropic* in anisotropic superconductors. For the calculation of such anisotropic moduli it is essential to know precisely the equilibrium structure of the FLL, otherwise they might turn out to be negative.

Also recently, Moore[4] extended Eilenberger's[5] treatment of the distorted FLL (or perturbed order parameter) and found for $\langle u^2\rangle$ an expression similar to (1) but with $\hat{c}_{44}$ replaced by the superfluid density ρ_s. One may write $\rho_s = 4\hat{c}_{44}k_h^2/k_{\rm BZ}^2$, where $k_h=(1-b)^{1/2}/\lambda$ is the reciprocal effective penetration depth and $k_{\rm BZ}=(2b)^{1/2}/\xi$ is the radius of the Brillouin zone (BZ) with $b=\bar{B}/B_{c2}(T)$ and $\xi=\lambda/\kappa$. Moore's result thus differs from Nelson's $\langle u^2\rangle$, Eq. (1), by a factor of $k_{\rm BZ}/2k_h=[b\kappa^2/2(1-b)]^{1/2}$. This factor may become large if $\kappa\gg 1$ and it diverges as $B_a\to B_{c2}(T)$. The resulting melting temperature is reduced by the same factor.

In this Letter I show that Nelson's and Moore's differing results follow in a natural way as approximations from the correct treatment which has to account for the nonlocal elastic response of the FLL which applies when $2b\kappa^2\approx \bar{B}\ln\kappa/B_{c1}$ is $\gg 1$.[6] (a) Nelson's $\langle u^2\rangle=\langle u^2\rangle_{\rm loc}$ follows in the *local* elastic approximation which replaces $c_{44}(k)$ by $c_{44}(0)=\hat{c}_{44}$ and thus considers only the (too large) stiffness of a homogeneous magnetic field. (b) Moore's $\langle u^2\rangle=\langle u^2\rangle_{\rm nl}$ follows in the extreme *nonlocal* limit which replaces $c_{44}(k)$ by $c_{44}k_h^2/k_\perp^2$; this is correct if $k_h<k_\perp<k_{\rm BZ}$ and $k_z<k_\perp$. This treatment considers only the (too small) stiffness of the vortex cores (or of the order parameter); it yields a diverging $c_{44}(k)$ for $k\to 0$ as discussed in parts I and IV of Ref. 6, where this unphysical divergence is removed. I will show that in the correct expression for $\langle u^2\rangle$, Eq. (5) below, the dominating contribution comes from large $k_\perp$ values; Moore's $\langle u^2\rangle$ is, therefore, close to the correct result.

I shall further show in this Letter that an estimate from the fluctuating shear strain yields much lower melting temperatures than does Lindemann's criterion and that T_m is reduced further if one goes beyond the commonly used elastic continuum approximation and if one takes short-wavelength compressional waves into account in $\langle u^2\rangle$.

The thermal average $\langle u^2\rangle$ may be calculated from the equipartition principle by ascribing to each eigenmode (in a finite volume V, with discrete $\mathbf{k}$) the average thermal energy $k_BT/2$. These eigenmodes are strongly overdamped and thus have no kinetic energy. Their elastic energy is $\frac{1}{2}u_\alpha(\mathbf{k})\phi_{\alpha\beta}(\mathbf{k})u_\beta(\mathbf{k})$, where $\phi_{\alpha\beta}(\mathbf{k})$ is the elastic matrix of the FLL ($\alpha,\beta=x,y$). This gives $\langle u_\alpha(\mathbf{k})u_\beta(\mathbf{k})\rangle = k_BT\phi_{\alpha\beta}^{-1}(\mathbf{k})$ and, after letting $V\to\infty$,

$$\langle u^2\rangle = k_BT\int (d^3k/8\pi^3)[\phi_{xx}^{-1}(\mathbf{k})+\phi_{yy}^{-1}(\mathbf{k})]\,, \qquad (2)$$

where the integration is over the BZ and over $-\infty<k_z<\infty$. The elastic matrix was derived in a series of four papers.[6] It is periodic in $\mathbf{k}$ space, but for many purposes may be replaced by its (isotropic) continuum approximation[7] which becomes exact for $k\ll k_{\mathrm{BZ}}$:

$$\phi_{\alpha\beta}^{\mathrm{cont}}(\mathbf{k}) = k_\alpha k_\beta[c_{11}(k)-c_{66}] + \delta_{\alpha\beta}[k_\perp^2 c_{66}+k_z^2c_{44}(k)]\,, \qquad (3)$$

where

$$c_{11}(k)=\hat{c}(1+k^2/k_h^2)^{-1}(1+k^2/k_\psi^2)^{-1}+c_{66}\,,$$
$$c_{66}\approx\hat{c}(1-b)^2(8b\kappa^2)^{-1}c_1c_2\,, \qquad (4)$$
$$c_{44}(k)=\hat{c}[(1+k^2/k_h^2)^{-1}+k_h^2/k_{\mathrm{BZ}}^2]\,,$$

with $\hat{c}=\bar{B}^2/\mu_0\approx\hat{c}_{44}$, $k_\psi^2=2\kappa^2k_h^2$, and $k^2=k_x^2+k_y^2+k_z^2=k_\perp^2+k_z^2$. The constants $c_1=1-(2\kappa^2)^{-1}$ and $c_2=1-0.58b+0.29b^2$ may be replaced by 1 for our purposes.

The moduli (4) apply when the vortex fields overlap such that each vortex interacts with many neighbors; if $\kappa\gg1$ this means $2/\kappa^2<b<1$ or $4B_{c1}/\ln\kappa<\bar{B}<B_{c2}$. Larkin and Ovchinnikov[8] have shown that these $c_{11}(k)$ and $c_{44}(k)$ apply not only within the Ginzburg-Landau theory (as in Ref. 6) but at all temperatures $0<T<T_c$. An approximate general expression for $c_{66}(T,B,\kappa)$ is given in Ref. 9. At very small inductions $b\lesssim1/2\kappa^2$, corresponding to $a\lesssim3.8\lambda$, the interaction with other than nearest neighbors may be disregarded and one gets $c_{11}(k)\approx c_{11}(0)\approx3c_{66}\sim\exp(-a/\lambda)$. I shall not write down these expressions since for such extremely small $\bar{B}\lesssim B_{c1}/\ln\kappa$ (or $B_a\approx B_{c1}$) the behavior of the FLL is dominated by even very weak pinning. This statement, concluded from the always present slight hysteresis of magnetization curves near B_{c1}, applies even to the purest Nb crystals available. In twin-boundary-free crystals of high-T_c superconductors the flux lines, due to their small core radius $\approx\xi$, are possibly pinned even by the atomic structure itself (a sort of Peierls potential).[10] Fascinating decoration experiments[7,11,12] may help to determine this pinning force.

Inserting (3) into (2) one gets

$$\langle u^2\rangle = \frac{k_BT}{2\pi^2}\int_0^{k_{\mathrm{BZ}}}dk_\perp k_\perp\int_0^{k_{\mathrm{BZ}}}dk_z\left[c_{66}k_\perp^2+\frac{\hat{c}_{44}k_z^2k_h^2}{k_h^2+k_\perp^2+k_z^2}\right]^{-1} \qquad (5)$$

plus a similar term with c_{66} replaced by $c_{11}(k)$; this term is commonly disregarded but gives a contribution of similar order as (5) since, for large k, $c_{11}(k)$ is very small; one has $c_{11}(k_{\mathrm{BZ}})\approx c_{66}$. The k-independent term of $c_{44}(k)$ [Eq. (4)] has been used to cut off the integration over k_z. It is interesting to note that this term, originating from the stiffness of the vortex cores, introduces the BZ radius k_{BZ} into the integration over k_z. This constant term follows from the limit $k\to0$ in part IV of Ref. 6 as

$$\bar{B}(B_a-\bar{B})/\mu_0\approx(B_{c2}-\bar{B})/2b\kappa^2\mu_0\approx\hat{c}_{44}k_h^2/k_{\mathrm{BZ}}^2$$

(Ref. 13).

A good approximation to the integral (5) is[14] [note that $k_h^2\hat{c}_{44}/k_{\mathrm{BZ}}^2c_{66}\approx4/(1-b)$ is always large]

$$\langle u^2\rangle\approx\langle u^2\rangle_{\mathrm{loc}}\left[\frac{2k_{\mathrm{BZ}}^2}{\pi k_h^2}\left(\frac{c_{66}}{\hat{c}_{44}}\right)^{1/2}+\left(1+\frac{k_{\mathrm{BZ}}^2}{4k_h^2}\right)^{1/2}\right]$$
$$\approx\langle u^2\rangle_{\mathrm{loc}}\left[\left(\frac{2b\kappa^2}{\pi^2}\right)^{1/2}+\left(1+\frac{b\kappa^2/2}{1-b}\right)^{1/2}\right]. \qquad (6)$$

In (6) the first term (from $k_z>k_\perp$) originates from the core stiffness, the second term (from $k_z<k_h$) is $\langle u^2\rangle_{\mathrm{loc}}$, and the third term (from $k_\perp>k_h$) is just Moore's result[4] (up to a factor $\approx\beta_A=1.16$). This main term, $\langle u^2\rangle_{\mathrm{nl}}$, exceeds $\langle u^2\rangle_{\mathrm{loc}}$ by a factor of $k_{\mathrm{BZ}}/2k_h$ as stated above. The dominating contribution to this nonlocal term comes from $k_\perp\lesssim k_{\mathrm{BZ}}$ where $c_{44}(k)\approx\hat{c}_{44}k_h^2/k_\perp^2$. The enhancement of $\langle u^2\rangle$ thus comes from the softening of the FLL with respect to periodic tilt waves with $\mathbf{k}$ vector *perpendicular* to the FL's (Fig. 1) and not from the reduced line tension of wavy FL's with increasing k_z. Keeping only this main term in (6), Lindeman's criterion $\langle u^2\rangle_{\mathrm{nl}}=c^2a^2$ yields a melting temperature

$$T_m=2k_B^{-1}c^2a^2n^{-1}k_h(c_{66}\hat{c}_{44})^{1/2}$$
$$\approx T_m^*(1-b)^{3/2}b^{-1/2}\lambda(0)/\lambda(T_m)\,, \qquad (7)$$

where

$$T_m^*=(2/3)^{1/2}c^2\phi_0^2/\mu_0k_B\lambda(0)\kappa\,, \qquad (8)$$

and $b=\bar{B}/B_{c2}(T_m)$. This T_m is smaller than the local elastic result by a factor of $[2(1-b)/b\kappa^2]^{1/2}$. For $c=0.1$, $\lambda(0)=2500$ Å, and $\kappa=200$, one gets $T_m^*=40$ K.

A different criterion for the melting of the FLL is that the thermal fluctuation of the shear strain γ of the FLL

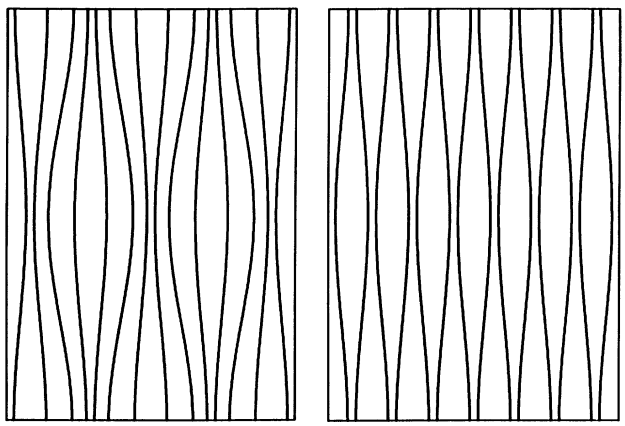

FIG. 1. Tilt modes of the vortex lattice with short wavelengths perpendicular to the vortices, $2\pi/k_\perp = 6a$ (left) and $2a$ (right). For not too small, constant k_z the elastic energy is dominated by the tilt energy which decreases with decreasing wavelength as $1/k_\perp^2$. The depicted modes give the dominating contribution to $\langle u^2\rangle$, Eq. (5), and in addition possess maximum probability to cause vortex cutting.

reaches a critical value γ_c. One has

$$\langle\gamma^2\rangle = k_B T \int (d^3k/8\pi^3)[k_y^2\phi_{xx}^{-1} - 2k_xk_y\phi_{xy}^{-1} + k_x^2\phi_{yy}^{-1}]\,, \tag{9}$$

$$\langle\gamma^2\rangle \approx k_B T \int (d^3k/8\pi^3)k_\perp^2[c_{66}k_\perp^2 + c_{44}(k)k_z^2]^{-1} \approx k_B T k_{\mathrm{BZ}}^4/16\pi k_h (c_{66}\hat{c}_{44})^{1/2} = \langle\gamma^2\rangle_{\mathrm{nl}}\,. \tag{10}$$

The criterion $\langle\gamma^2\rangle_{\mathrm{nl}}^{1/2} = \gamma_c$ reproduces the T_m [Eq. (7)] if one puts $\gamma_c = 2\pi^{1/2}3^{-1/4}c = 0.27$. This value of γ_c appears rather high. The critical shear stress required to deform an atomic or FLL plastically, $\tau_c \approx c_{66}/30$,[15] suggests a smaller value, $\gamma_c \approx \frac{1}{30}$, which would reduce the melting temperature (7) by a factor of 65. A somewhat larger value $\gamma_c \approx \frac{1}{20}$ to $\frac{1}{10}$ appears more realistic since the fluctuating shear is not uniform. Temperature-induced local plastic deformations (small dislocation loops) possibly heal (the loops collapse) if they stay below some threshold concentration, amplitude, or extension. Stable defects may result when FL's cut and reconnect, a process requiring little activation energy.[16] Temperature-induced disorder may set in when some of the fluctuating FL's touch and cut irreversibly. Smaller values $c \approx \frac{1}{20}$ appear thus more realistic in the Lindemann criterion. This and the shear-strain argument hint at a considerably smaller T_m^* value, ≈ 10 K for the above example.

Larger $\langle u^2\rangle$ and $\langle\gamma^2\rangle$ and thus lower T_m are also expected from the following arguments:

(a) At $b > 0.5$ the factor c_2 in c_{66} [Eq. (4)] reduces T_m^* by a factor of 0.84.

(b) The term containing $c_{11}(k)$ gives a contribution $\approx (2/3\pi)\langle u^2\rangle_{\mathrm{nl}}$ to $\langle u^2\rangle$ and thus reduces T_m^* by a factor of 0.82. There is no such term in $\langle\gamma^2\rangle$, Eq. (9).

(c) A reduction by nearly a factor of 2 results when in (2) or (9) the correct elastic matrix is inserted. For k near the BZ boundary the correct $\phi_{\alpha\beta}(\mathbf{k})$ is highly anisotropic. (This should not be confused with the material anisotropy which I do not consider here.) $\phi_{\alpha\beta}$ is obtained by replacing in the continuum approximation (3) the term $k_\alpha k_\beta c_{66}$ by a sum over reciprocal-lattice vectors $\mathbf{K}$ and $\mathbf{Q}$,[6,7]

$$\hat{c}\sum_{\mathbf{Q}\neq 0}\frac{k_h^2k_\psi^2 - k_h^4}{(\mathbf{Q}-\mathbf{k})^2 + k_h^2}\sum_{\mathbf{K}}\alpha_{\mathbf{K}}\frac{(\mathbf{Q}-\mathbf{k})_\alpha(\mathbf{Q}-\mathbf{K}-\mathbf{k})_\beta}{(\mathbf{Q}-\mathbf{K}-\mathbf{k})^2 + k_\psi^2}\,, \tag{11}$$

plus a constant which follows from $\phi_{\alpha\beta}(0) = 0$. The constants $\alpha_{\mathbf{K}} = \alpha_{mn}$ are $\alpha_{00} = 1$, $\alpha_{10} = -0.3032$, $\alpha_{11} = 0.0804$, $\alpha_{20} = 0.0525$, etc.[6] The expression (11) follows from the Ginzburg-Landau theory for $b > 0.6$. If one puts $\alpha_{\mathbf{K}} = \delta_{\mathbf{K},0}$ then (11) applies to $b < 0.25$ and approximately to $0 < b < 1$. The anisotropic $c_{66}(\mathbf{k})$ at the BZ boundary (for $\mathbf{k} = \mathbf{K}_{10}/2$ and $\mathbf{k} = \mathbf{K}_{11}/2$) takes the values $\phi_{yy}(k,0,0,)/k^2 \approx 0.5c_{66}$ and $\phi_{xx}(0,k,0)/k^2 \approx (1.2\text{-}3.1)c_{66}$ (depending on b), cf. Figs. I1 and II2 in Ref. 6. Since c_{66} appears in the denominators in (5) and (10), and since mainly $\mathbf{k}_\perp$ values near the BZ boundary contribute to these integrals, this correction approximately doubles $\langle u^2\rangle$ and $\langle\gamma^2\rangle$.

(d) The elastic response of the FLL to a shear stress becomes nonlinear (softens) at rather small strains $\gamma \approx 0.02$ to 0.03.

(e) When plastic deformation sets in the FLL softens further, mainly because screw dislocations[17,18] (oriented perpendicular to the FL's) can move freely along the FL's (they feel no Peierls potential).

All these effects tend to decrease T_m.

Nonlinear elastic effects and the behavior of an entangled or melted FL arrangement in principle may be calculated, at all inductions $0 < \bar{B} < B_{c2}(T)$, from the interaction $V_3(\mathbf{r}_\perp,z;\mathbf{r}_\perp',z')$ between vortex line elements given in Ref. 18. Nelson's[2,3] model interaction $V_2(\mathbf{r}_\perp;\mathbf{r}_\perp')$ is the local approximation to this general interaction and applies if $k_z < k_h$, i.e., if the bending wavelength of the FL's, $2\pi/k_z$, exceeds the rather large length $2\pi\lambda/(1-b)^{1/2}$.

Strong pins are expected to stabilize the FLL and to increase T_m. Weak pins should not change the above discussion of T_m. Depinning of weak pins of range $\approx\xi$ should occur when $\langle u^2\rangle \approx \xi^2$, i.e., at temperatures below T_m if $\xi < ca$ and at $T \approx T_m$ if $\xi > ca$, corresponding to $b > 7c$. Experiments involving vortex-pin interactions are thus not specific for the determination of T_m; they might rather yield the onset of thermally assisted flux flow[19] (for small driving force) or flux creep.[20] FLL melting should be investigated by methods which determine the FLL structure directly, like small-angle neutron scattering or muon-spin rotation (μ^+SR). Recently the μ^+SR signal (field density) has been calculated for perturbed vortex lattices.[21]

Note added.—After this work was submitted I received a preprint by Houghton, Pelcovits, and Sudbø,[22] who use nonlocal elasticity to derive a similar result for $\langle u^2 \rangle$, which partly accounts for anisotropy.

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