Black-Hole Decay and Topological Stability in Quantum Gravity

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We show that black holes are topologically stable under quantum fluctuations but unstable under quantum emission and absorption of gravitons. The probability of emission behaves as $\exp[-\alpha(M_f-M_i)]$, where M_i and M_f are the masses associated with the initial and final states, and α is a positive constant of the order of 1. As the black hole loses mass it evolves towards a state corresponding to a black hole of very small mass that cannot be distinguished from a pure graviton state.

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A consistent theory of quantum gravity should at least give an answer to the following problems[1-3]: (1) dynamical fluctuations of the spacetime topology; (2) quantum decay of black holes (BH's); and (3) singularities in general relativity. Classically, the action is stationary with respect to local variations of the geometry, in the neighborhood of a solution of Einstein's equations with a fixed topology. Quantum mechanically, however, there is no reason for the geometry not to tunnel between two distinct topological configurations,[4-9] in spite of the fact that changing the topology may imply a singular transformation in the spacetime manifold. This means that the system might have to go through an infinite energy or action barrier.

On the other hand, it is generally accepted that although BH's are classically stable, they can decay via emission of quantum-mechanical particles.[10] It must be emphasized that this effect is inferred from a semiclassical approximation where the gravitational field is classical. If the gravitational field itself were to be quantized, one should also expect a BH to decay into gravitons. To the best of our knowledge no suitable model has been proposed to analyze this issue. The basic obstacle in this case is the fact that the number of degrees of freedom in the BH geometry is so restricted by the symmetries imposed on the system that no process can take place.[11] This could be circumvented by the introduction of new dynamical variables enlarging the phase space containing the BH geometry.

In the context of the Wheeler-DeWitt approach to quantum gravity,[1,2] we examine the above-mentioned issues for geometries that in principle allow for topological transitions and quantum decay of BH's.

Singularities are predicted in classical gravity to occur in two situations: At the beginning of the present expansion of the Universe (big bang) and in the collapse of isolated regions of high-mass concentration (BH's).[12] It has been widely discussed whether quantum effects can avoid singularities in cosmological solutions.[3,13] In the BH case it is not known whether the singularity will eventually disappear or will stabilize in a mini-BH of finite size. This question is clearly related to the quantum decay of BH's and it will be treated in the model proposed here.

Let us consider the Schwarzschild-type geometries described by

$$ds^2=N^2(r,t)dt^2-\Delta^{-2}(r,t)dr^2-r^2K^2(\theta,t)(d\theta^2+\sin^2\theta\, d\varphi^2)\,, \qquad (1)$$

where $\Delta(r,t)=[\lambda(t)-2M/r]^{1/2}$, with M a constant. This corresponds to the simplest extension of the static BH geometry which allows for dynamical fluctuations sufficient in the quantum version of the theory, topological changes, and evolution of BH's. In general, λ and K are taken as time-dependent functions. For particular choices of N, λ and K, the metric (1) describes classical solutions. If $N=\Delta$ and λ is a constant, the case $\lambda>0$, $K=1\sqrt{\lambda}$ is a Schwarzschild BH; $\lambda=0$, $K=1/\sin\theta$ is a Kasner universe; $\lambda<0$, $K=\tanh\theta/\sqrt{-\lambda}$ a Bianchi III universe. In particular, we note that for different values of λ these configurations have different topologies, at least in the cases for which the classical solutions exist.

The Hilbert-Einstein action for the metric (1) reduces[14] to Arnowitt-Deser-Misner action[15,16]

$$S=4\pi a_0\int\left[\frac{2r^2}{N}\left(\frac{\dot K^2}{\Delta}-\frac{\dot\lambda \dot K K}{\Delta^3}\right)+\frac{N}{\Delta}(\lambda K^2-1)\right]dr\,dt+\text{surface terms}\,. \qquad (2)$$

Clearly, λ and K are the only dynamical degrees of freedom and N is a Lagrange multiplier. Starting from action (2) above, one can calculate the canonical momenta π_λ and π_K, and construct the Hamiltonian of the theory. Since N is the

Lagrange multiplier, its canonical momentum is a primary constraint $\pi_N \approx 0$. The preservation in time of this constraint, in turn, means that the Hamiltonian is itself a constraint,[17]

$$\int N\mathcal{H}\, dr \approx 0\,, \tag{3}$$

where

$$\mathcal{H} = -\frac{\Delta^3}{r^2}\left[\frac{1}{K}\pi_K\pi_\lambda + \frac{\Delta^2}{K^2}\pi_\lambda^2\right] - \frac{1}{\Delta}(\lambda K^2 - 1) + \text{surface terms}\,. \tag{4}$$

We remark that for any choice of N, a classical configuration must satisfy the constraint (3). Moreover, in the limit $r \to \infty$,

$$\lambda K^2 - 1 = 0\,, \tag{5}$$

and since this must hold for any r, Eq. (5) is a necessary condition for classical configurations.[18] In what follows, we opt for the gauge $N=\Delta$, which makes the contribution of the surface terms at $r=2M/\lambda$ and at $r\to\infty$ cancel each other.[19]

According the Dirac's program,[20] quantum theory is constructed by replacing the canonical momenta by $\hat{\pi}_\lambda = -i\,\partial/\partial\lambda$, $\hat{\pi}_K = -i\,\partial/\partial K$, and the constraint (3) by the restriction on the Hilbert space of wave functions $\psi(\lambda,K)$,

$$\hat{H}\psi = \left[\frac{\lambda^2}{4K^2}\left(\lambda\frac{\partial}{\partial\lambda}\lambda\frac{\partial}{\partial\lambda} + \frac{4}{3}K\frac{\partial}{\partial K}\lambda\frac{\partial}{\partial\lambda}\right) + \frac{4M^2}{\lambda}(\lambda K^2 - 1)\right]\psi(\lambda,K) = 0\,. \tag{6}$$

The Wheeler-DeWitt equation (6) describes the evolution of the amplitude $\psi(\lambda,K)$ on the minisuperspace of metrics (1). The order for the operators in (6) was chosen so that the kinetic term would be the Laplace-Beltrami operator on the minisuperspace.[21] We have also subtracted an infinite constant from $\hat{H}$ which corresponds to the infrared contributions to the zero-point energy, and which is identically zero in the case $\lambda K^2 - 1 = 0$. We note that the boundary $r=2M/\lambda$, $\lambda > 0$, of the spacetime manifold arises from the need to have a sensible classical theory as a limit, namely, the Hamiltonian density (4) as the generator of time evolution. However, in the quantum regime the restriction $\lambda > 0$ is no longer necessary and the possibility of quantum fluctuations of the signature and tunneling between different topologies is not excluded. Introducing new coordinates (σ,μ) in superspace through

$$\lambda = e^\sigma,\quad K = e^{2(\mu+\sigma)/3}\,, \tag{7}$$

Eq. (6) reads

$$\left[\left(\frac{\partial^2}{\partial\mu^2} - \frac{\partial^2}{\partial\sigma^2}\right) + U(\mu,\sigma)\right]\psi(\mu,\sigma) = 0\,, \tag{8}$$

where

$$U(\mu,\sigma) = 16M^2(e^{4\mu/3 - 5\sigma/3} - e^{8\mu/3 + 2\sigma/3})\,. \tag{9}$$

The above choice of coordinates is appropriate for the BH sector, $\lambda > 0$. A topological transition would be signaled by the probability density approaching a nonzero value as $\lambda \to 0$.

The hyperbolic character[22] of Eq. (8) has several consequences: (i) The minisuperspace has a "light-cone" structure; (ii) the probability density is not positive definite, and the one-particle interpretation is not fully consistent (Klein's paradox[23]). The light-cone structure will be used to interpret $i\,\partial/\partial\sigma$ and $i\,\partial/\partial\mu$ as energy and momentum operators for the wave solutions of Eq. (8). In spite of (ii), the one-particle interpretation will be useful for the analysis of topological stability but for the discussion of BH decay one may adopt the point of view of second quantization.

From (9) we observe that the potential U is analytical throughout the plane (σ,μ). U vanishes on the line $4\mu + 7\sigma = 0$, that corresponds to the classical configurations (5). Above this line U is negative and it approaches $-\infty$ for $\mu\to\infty$ or $\sigma\to\infty$, and below this line $U\to 0^+$ as $\mu\to-\infty$, and $U\to+\infty$ as $\sigma\to-\infty$. We note that the points on the line $U=0$ correspond to classical Schwarzschild BH's of masses

$$M_{\mathrm{BH}} = M\exp(-\sigma/2) \tag{10}$$

decreasing monotonically with σ (see Fig. 1). The solutions of (8) can be chosen so that in the region $\mu\to-\infty$ they form a complete set of plane waves. There we can prepare a wave packet and see its evolution in the μ direction, which plays the role of time for $U>0$. Actually, when $\mu<0$ and $U>0$, the potential is a slowly varying function of μ and one can approximately write $\psi \sim e^{-i\epsilon\mu}\varphi_\epsilon(\sigma)$, where $\varphi_\epsilon(\sigma)$ satisfies the Schrödinger equation

$$[-\partial^2/\partial\sigma^2 + U(\sigma)]\varphi_\epsilon(\sigma) = \epsilon^2\varphi_\epsilon(\sigma)\,.$$

The potential $U(\sigma)$ produces a strong damping of the wave functions $\varphi_\epsilon(\sigma)$ as $\sigma\to-\infty$, acting as an infinite barrier. This means that in the eikonal approximation a wave packet centered somewhere in this region will be driven to the region $U<0$ as μ increases, necessarily crossing the line of classical configurations $U=0$. The infinite potential barrier at $\sigma\to-\infty$ strongly suppresses the possibility of transition of the system to configurations with $\lambda\le 0$. Therefore BH's are topologically stable under quantum fluctuations that keep the geometry within the class (1).

For $U<0$ the roles of space and time are interchanged

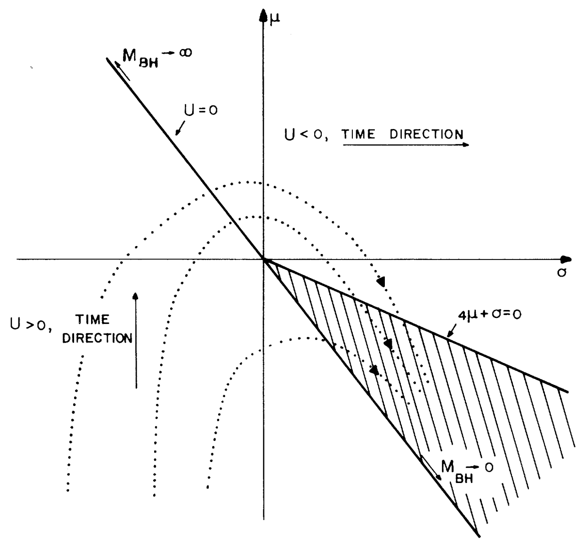

FIG. 1. The (σ,μ) minisuperspace. The line $U=0$ contains all the classical BH states of decreasing masses as $\sigma\rightarrow\infty$. The dotted curves are typical classical trajectories associated to Eq. (8). All these trajectories end up in the hatched region.

between μ and σ, and a similar analysis indicates that as $4\mu+\sigma\rightarrow\infty$ the potential becomes infinitely repulsive. Thus the future of any wave packet lies in the region $\sigma\rightarrow+\infty$ and $\mu\rightarrow-\infty$, with $4\mu+\sigma<0$, in the eikonal approximation.

In view of this we can assert that a classical BH is unstable, in the sense that any wave packet centered on the line of classical configurations, $U=0$, will be inevitably driven away by quantum effects. Perturbatively these quantum states should appear as a BH plus gravitons. Nevertheless, the perturbative regime cannot be maintained indefinitely, as the tendency is to evolve into a configuration dominated by quantum effects. On the other hand, any state prepared in the region $U>0$ will evolve towards the line $U=0$.

In a neighborhood of $U=0$, quantum processes can be analyzed in a perturbative scheme. For example, a transition from a state below the line $U=0$ to a point on the line can be interpreted as the absorption of gravitons by a BH, increasing its mass. Also, a transition from a state on the line $U=0$ to a point in a neighborhood above the line can be interpreted as the emission of gravitons by a BH, decreasing its mass. These processes are schematically described in Fig. 2. One can calculate the relative probability of these processes in a semiclassical approximation by evaluating the action along a path[24] going from $U>0$ to $U<0$. Taking the classical path tangent to the straight line $7\mu+4\sigma=0$ [orthogonal to $U=0$ in the metric (1,-1)], we estimate the relative probability to be

$$\frac{P(M_0\rightarrow M_2+g)}{P(M_1+g\rightarrow M_0)}\sim\exp[-\alpha(M_2-M_1)]\,, \qquad (11)$$

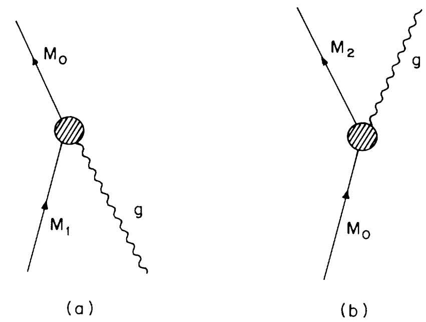

FIG. 2. (a) Absorption and (b) emission of gravitons by a BH.

where α is a positive constant of the order of 1. The masses M_1 and M_2 are calculated using the formula $M_i=M\exp(-\sigma_i/2)$ [cf. Eq. (10)] which is not strictly valid off the line, but is a good approximation for configurations close to the classical ones. From (11) we see that the probability for emission is greater than the probability for absorption of gravitons by BH's. This result goes in the same direction as Hawking's conclusion[10] about BH radiance, in the sense that the probability of emission increases as the BH loses mass. The above estimate does not furnish information on the distribution function of the energy of gravitons; this issue will be discussed in a forthcoming paper.

At this point we would like to call the attention to the nature of the line $U=0$. Let us consider a path reaching that line from below, where μ plays the role of time. If the path goes in the direction of *increasing* μ, it can be viewed as the absorption of a graviton by the BH. Analogously, a path in the direction of *decreasing* μ corresponds to the emission of a negative-energy graviton, the BH reaching a final state on the classical line $U=0$ with a smaller mass. By the time reversal, the latter process can be reinterpreted as the absorption of a graviton by a BH in the direction of *increasing* μ. By the same token all processes corresponding to paths starting from the classical line into the region $U<0$ can be interpreted as emission of gravitons by BH's. In this region σ plays the role of time, and for increasing σ the final state of the BH has a smaller mass. It is worth emphasizing that the line of classical states, $U=0$, acts as a divide between emission and absorption processes. Furthermore, "space" and "time" have their roles interchanged as the line is crossed. In this sense, for quantum processes, $U=0$ behaves similarly to the Schwarzschild horizon.

Finally, some comments should be made on the state that this system will approach as it evolves. In view of the unstable nature of BH's in the present scheme, a classical BH will inevitably approach a configuration with a large number of gravitons produced at the expense of its mass. Moreover, since any localized wave packet will eventually spread out under the influence of the potential, the final state will necessarily be a superposition of BH states of very small masses plus gravitons. The effective coupling for the vertex of Fig. 2 is $|U|$ and it approaches zero as the system evolves. Thus the decay processes will fade out, producing a configuration of noninteracting BH's and gravitons. From the semiclassical approximation (11), the absorption and emission probabilities are expected to approach zero at the same rate, as $M_{\rm BH} \rightarrow 0$. Also, as $U \rightarrow 0$, it becomes impossible to distinguish between a state of a BH of very small mass and a pure graviton state. To sum up, the outcome of the process will be a state of noninteracting, indistinguishable gravitons and mini-BH's.

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[1]B. S. DeWitt, Phys. Rev. **160**, 1113 (1967).

[2]J. A. Wheeler, in *Relativity, Groups and Topology,* edited by B. DeWitt and C. DeWitt (Gordon and Breach, New York, 1964); in *Battelle Recontres,* edited by C. DeWitt and J. A. Wheeler, (Benjamin, New York, 1968).

[3]J. B. Hartle and S. W. Hawking, Phys. Rev. D **28**, 2960 (1983).

[4]B. DeWitt, in *Proceedings of the Third Seminar on Quantum Gravity,* edited by M. A. Marzov, V. A. Beregin, and V. P. Frolov (World Scientific, Singapore, 1985).

[5]A. Anderson and B. DeWitt, Found. Phys. **16**, 91 (1986).

[6]M. Castagnino, Phys. Rev. D (to be published).

[7]S. W. Hawking and R. Laflamme, University of Cambridge, Department of Applied Mathematics and Theoretical Physics, Report No. DAMTP/R-88/3, 1988 (to be published).

[8]S. Coleman, Harvard University Report No. HUTP-88/A008, 1988 (to be published).

[9]A. Vilenkin, Phys. Rev. D **37**, 888 (1988).

[10]S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).

[11]F. Lund, Phys. Rev. D **8**, 3247 (1973).

[12]S. W. Hawking and R. Penrose, Proc. Roy. Soc. London A **314**, 529 (1970).

[13]A. D. Sakharov, Dokl. Akad. Nauk SSSR **177**, 70 (1967) [Sov. Phys. Dokl. **12**, 1040 (1968)].

[14]We use units such that $G=\hbar=c=1$. We define the constant $a_0=\int \sin\theta\, d\theta\, d\varphi$; it is finite for the cases $\lambda>0$. The θ dependence of $K(\theta,t)$ can be gauged away, and we choose $dK/d\theta=0$. We drop from the Lagrangian the multiplicative constant $4\pi a_0$. An overdot means derivative with respect to t.

[15]R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation: An Introduction to Current Research,* edited by L. Witten (Wiley, New York 1962).

[16]T. Regge and C. Teitelboim, Ann. Phys. **88**, 286 (1974).

[17]P. A. M. Dirac, Proc. Roy. Soc. London A **245**, 333 (1958); Phys. Rev. **114**, 924 (1959).

[18]Condition (5) should not be regarded as a constraint, on an equal footing as (3), for example. It characterizes a feature of the classical solutions only and cannot therefore be expected to hold for quantum processes.

[19]Now there is no gauge freedom left: The coordinates are chosen so that g_{00} and g_{0i} are fixed functions. What remains is the freedom to make global time translations generated by $H=\int N\mathcal{H}\, d^3x$. This is analogous to setting $dx^0/d\tau=N=$const for the relativistic particle, in which case the only freedom left is that for rigid time displacements of the entire worldline. See, for example, C. Teitelboim, Phys. Rev. D **28**, 297 (1983).

[20]P. A. M. Dirac, *Lectures on Quantum Mechanics* (Belfer Graduate School of Science, Yeshiva Univ. Press, New York, 1964).

[21]T. Christodoulakis and J. Zanelli, Classical Quantum Gravity **4**, 851 (1987).

[22]This is due to the fact that λ and K characterize a submanifold of the superspace that contains a "timelike" trajectory (relative to DeWitt's supermetric) corresponding to conformal transformations of the spatial components of geometry (1). See, for example, Ref. 1.

[23]G. Baym, *Lectures on Quantum Mechanics* (Benjamin, New York, 1969).

[24]R. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).