Interference of Directed Paths in Disordered Systems

Ernesto Medina and Mehran Kardar
Physics Department, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Yonathan Shapir and Xiang Rong Wang
Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627
(Received 6 July 1988)

We consider sums over directed paths, interfering due to quenched random elements controlling either the phase or the sign of each term. The former enter the determination of electronic transport in the localized regime, and the latter that of correlations in the high-temperature phase of the Ising spin-glass. Numerical computations in two dimensions indicate anomalous fluctuations dominated by untypical paths, with scaling behavior similar to that of random but positive impurities. This is argued to follow from a bound state, due to a subtle attraction between pairs, in replica space.

PACS numbers: 71.55.Jv, 05.40.+j, 72.20.Dp, 75.10.Nr

In many circumstances, an interesting physical quantity is obtained as a sum of contributions over different "paths." If terms in the sum have different signs (or phases), complicated interference patterns can arise. What happens when quenched impurities modify the sign (phase) of each contribution? Here we examine two such cases: One is the high-temperature series for the two-spin correlation function in the Edwards-Anderson[1] model of a spin-glass.[2] Each term is a product over bonds along different paths connecting the two spins, and as the bonds are random in sign, each contribution may be positive or negative. Our second example concerns the quantum interference between tunneling paths, resulting from multiple elastic scattering by impurities. We start with a simplified model introduced by Nguen, Spivak, and Shklovskii[3] (NSS), and generalize it to include random tunneling events represented by random phases. What are the statistical properties of such sums, and where do the dominant contributions come from?

A similar problem appears for directed polymers in random media.[4] The terms in the partition function Z are Boltzmann weights of polymer configurations, and impurities produce a randomness in magnitude, rather than phase, of each term. For strong randomness, $\ln Z$ has a well-defined distribution, its mean increasing linearly with the polymer length t, and fluctuations scaling as t^{ω}. Transverse fluctuations of typical paths scale as t^{ν} with $\nu > \frac{1}{2}$. In $d=2$, it can be shown by the replica method,[5] or by mapping to a Burgers' equation,[6] that $\omega = \frac{1}{3}$ and $\nu = \frac{2}{3}$. With impurities producing randomness in sign (phase), rather than in amplitude, we may expect wilder fluctuations in $Z(t)$. Here we present numerical results in two dimensions, and analytic arguments, suggesting that the asymptotic behavior with random signs or phases is in fact similar to that of random amplitudes.[4] Numerical results again show that $\ln Z(t)$ has a mean proportional to t, and a standard deviation that grows as t^{ω} with $\omega = 0.33 \pm 0.05$ (a replica argument suggests $\omega = \frac{1}{3}$ exactly). The numerical results for transverse fluctuations are less clear-cut, but the estimate $\nu = 0.68 \pm 0.05$ is in agreement (though somewhat larger) with the theoretical value of $\frac{2}{3}$. We suggest that in higher dimensions random phases and random magnitudes may again lead to similar behaviors for directed paths.

Consider the Ising spin-glass (Hamiltonian $H = \sum_{\langle ij \rangle} J_{ij} \sigma_i \sigma_j$) with *nearest-neighbor* bonds J_{ij} equal to $+J$ or $-J$ with the same probability.[1,2] The two-spin correlation function $[\sigma_0 \sigma_r]_{\rm av}$, in a high-temperature expansion, is a sum over paths connecting σ_0 and σ_r, each contributing a product of $\tanh(\beta J_{ij})$ factors along its length ($[\cdots]_{\rm av}$ denotes thermal averaging). In the limit $\tanh(\beta J) \to 0$, only the shortest (directed) paths connecting the end points need to be considered. Figure 1 indicates three such paths in the series for spins separated diagonally on a square lattice. Since all these direct-

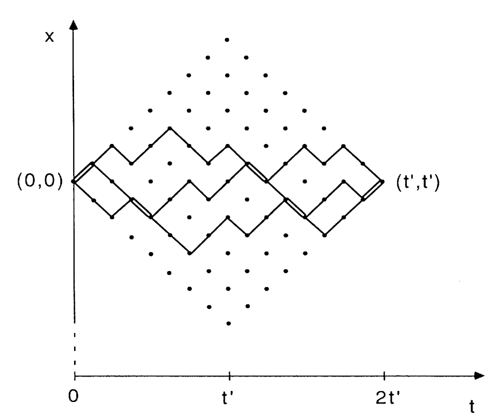

FIG. 1. Directed paths of length $t = 2t'$ connecting two diagonally separated points on a square lattice. There are random elements on bonds or sites.

ed paths are of the same length,

$$[\sigma_{0,0}\sigma_{t',t'}]_{\rm av}=\tanh(\beta J)^{2t'}Z(2t')\,,$$

where $Z(2t')$ is the sum over the $N=(2t')!/(t'!)^2$ possible paths, each contributing $+1$ or -1 depending on the product of signs of random bonds crossed. Statistical properties of $Z(t)$ were explored numerically by considering 2000 realizations for values of t up to 2000 ($t=2t'$ is the path length). Although the number of paths grows exponentially with t, a transfer-matrix algorithm allows exact calculation of $Z(t)$ in polynomial time.[4] For several choices of t we checked that positive and negative values of Z occur with equal probability. Moreover, histograms of $\ln|Z(t)|$ are well fitted by Gaussian forms. The mean of the distribution increases linearly with t [$\langle \ln|Z|\rangle=(0.32\pm0.01)t$], and the standard deviation grows as t^{ω} ($\langle\cdots\rangle$ denotes impurity averaging). Figure 2 is a log-log plot of $(\langle \ln|Z|^2\rangle-\langle\ln|Z|\rangle^2)^{1/2}$ vs t and can be fitted by a straight line with slope $\omega=0.33\pm0.05$.

Analytic information about the probability distribution can be extracted[5] by examining the moments $\langle Z^n\rangle$; this is related to the characteristic function for $\ln|Z(t)|$. As each term in Z describes a path traversing the random medium, the terms in Z^n correspond to the product of contributions from n independent paths. (For example, the three paths in Fig. 1 can be regarded as a term in Z^3.) Upon averaging, if m paths cross a particular bond ($0\le m\le n$), we obtain a factor of $[1+(-1)^m]/2$, which is 0 or 1 depending on the parity of m. For odd n there must be bonds with m odd, and hence $\langle Z^{2n+1}\rangle=0$, which of course implies and follows from the symmetry $P(Z)=P(-Z)$. For even moments $\langle Z^{2n}\rangle$, the only configurations that survive averaging are those in which the $2n$ replicated paths are arranged such that each bond is crossed an even number of times.

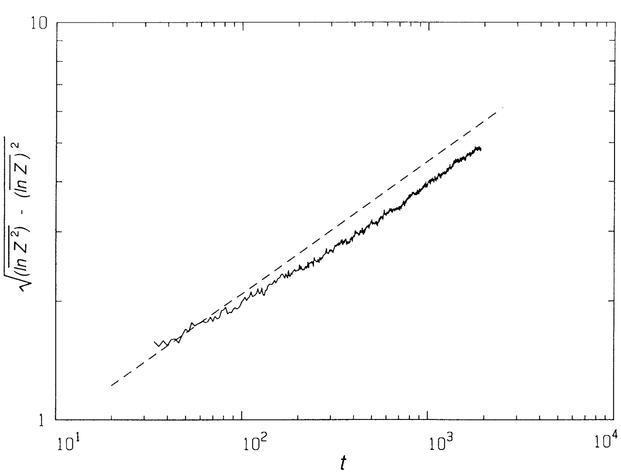

FIG. 2. The standard deviation of the logarithm of the sum over directed paths vs the path length t. The dashed line has slope $\omega=\frac{1}{3}$. (Here, impurity averaging is denoted by the overbar.)

These configurations correspond to drawing n independent paths between the end points and assigning two replica indices to each (Fig. 1 can now represent a term in $\langle Z^6\rangle$). However, there is a subtlety in calculating $\langle Z^{2n}\rangle$ from the n "double" paths: After two such paths cross, the outgoing paths can either carry the same replica pair labels as the ingoing ones, or they can exchange one label [e.g. (12),(34) $\rightarrow$ (12),(34) or (13),(24)]. Therefore there is a multiplicity of two per crossing of paths, which can be regarded as an attraction induced by an exchange of a replica partner.

Calculating $\langle Z^{2n}\rangle$ is now reduced to finding the sum over n paths attracted to each other by a factor of 2 per crossing. This is most easily evaluated in the continuum limit by regarding the paths as world lines of n attracting particles in one dimension. The ground state that dominates the long-t statistics is an n-body bound state with energy[5] $\epsilon_n=-\alpha n(n^2-1)$, where α is a positive constant. Hence, for large t, $\langle Z(t)^{2n}\rangle \sim 2^{nt}\exp[\alpha n(n^2-1)\times t]$. This is the characteristic function for $\ln(Z^2)$, and the absence of an n^2 term in the exponent indicates no second cumulant at order of t. The n^3 term reflects a third cumulant scaling[5] as t, and therefore the scale of fluctuations in $\ln|Z|$ is set by $t^{1/3}$; i.e., $\omega=\frac{1}{3}$ in agreement with numerical simulations. Note that the above arguments are almost identical to those presented for the random-magnitude problem.[5] The conceptual difference is in the origin of the attraction between paths which in this case is purely a statistical exchange effect. Note that we have ignored the multiplicities involved when three or more paths cross. Indeed the resulting multiplicity for multiple crossings cannot be written as a sum of attractive pair potentials. This, however, does not modify the above argument as in the continuum limit the probability of three particles at the same site is zero. Thus the two-particle attraction is sufficient to obtain the properties of the bound state.

In calculating the Mott variable-range-hopping conductivity, one needs the probability for an electron to hop between sites far apart. In many cases, this is not a unique tunneling event, but a multiple scattering from impurities (donors) between the initial and final states. NSS introduced[3] an Anderson tight-binding model ($H=\sum_i\epsilon_i a_i^\dagger a_i+V\sum_{\langle ij\rangle}a_i^\dagger a_j$), on a square lattice as in Fig. 1, to study the outcome of such multiple coherent tunneling events. The on-site energies ϵ_i are independent random variables taking values $+W$ with probability $1-p$, and $-W$ with probability p. Conductivity between the initial and final sites [(0,0) and (t',t')] is proportional to $|I|^2$, where I is the matrix element connecting these sites. As each step along a path joining the end points results in a factor of V/W, in the limit $V\ll W$, the main contribution comes from the N directed paths that exclude backward steps.[3,7,8] Hence

$$I=V\left(\frac{V}{W}\right)^{t-1}J(t)\ \text{ with }\ J(t)=\sum_{\alpha=1}^{N}\prod_{i_\alpha=1}^{t-1}\eta_{i\alpha}\,,$$

where for each path α, the product is over factors $\eta_i = W/\epsilon_i = \pm 1$ along its length. Although the random signs now originate at the sites, rather than at the bonds, in the asymptotic limit, this difference between the sums $Z(t)$ and $J(t)$ is not important.

The NSS model has received considerable attention,[7,8] but most analytic approaches have ignored the correlation between paths.[3,8] As demonstrated by Shapir and Wang,[7] such correlations are essential to the proper understanding of interference effects. Indeed for $p=\frac{1}{2}$, the sums $J(t)$ and $Z(t)$ are asymptotically identical, and the correlations between paths lead to the logarithmic distributions discussed earlier. Extending the replica analysis to $\langle J^n(t)\rangle$ with general p, upon averaging we obtain a factor of $1-2p$ for an odd number of paths through a particular site. Although for $p\neq\frac{1}{2}$ double occupation is no longer absolutely necessary, the reduced factor of $1-2p$ for $p\neq 0$ encourages paths to occur in pairs. The exchange attraction between pairs again leads to an n-body bound state. Thus we expect $\langle J^n(t)\rangle = \exp[\alpha_{\pm}(p)nt+\beta(p)n^3t]$, with different first cumulants α_+ and α_- for even and odd moments. NSS suggest[3] a transition in the behavior of $P(J)$ at $p=p_c\sim 0.05$: For $p<p_c$ J is mostly positive, while for $p>p_c$, positive and negative values occur with equal probability. The above analysis indicates that if such a transition is present, it is not reflected in the behavior of moments of J.

As the electron wave function is a complex number, and in the most general hopping problem random tunneling elements are also present, we are led to consider random phases. In the continuum limit, interference effects result in a stationary wave function $\psi(x,t)$ satisfying[5]

$$\partial\psi/\partial t = \nu\nabla^2\psi + i\theta(x,t)\psi. \tag{1}$$

This is a "Schrödinger-type" equation in a random potential $\theta(x,t)$, except that t and x are both spatial coordinates (t is parallel to the hopping direction, and x is perpendicular to it). The crucial difference is that the "kinetic energy" term is real. This is because we are dealing with decaying (tunneling) waves, rather than propagating ones. It also provides a justification for ignoring backscattering paths which are essential to localization.[8] Equation (1) with real noise has been studied by renormalization-group methods.[4,6] A similar procedure for imaginary randomness indicates that it is a relevant operator, leading to renormalization-group flows towards strong coupling, but without a stable fixed point. A discrete version of Eq. (1) is obtained by assigning a random factor of $e^{i\theta}$ to each bond. Because of the arbitrariness in phase, all moments $\langle\psi^n\rangle$ of the wave function vanish, and we should examine the norms $W(x,t) = |\psi(x,t)|^2$. The moment $\langle W^n\rangle = \langle(\psi^*)^n(\psi)^n\rangle$ is obtained by looking at $2n$ replicated paths (n for ψ and n for ψ^*). On each bond crossed by m paths from ψ, and m' paths from ψ^*, the averaging $\{\langle\exp[i\theta(m-m')]\rangle = \delta_{m,m'}\}$ forces $m=m'$. Thus again the original $2n$ paths coalesce to form paths for n pairs. The same factor-of-2 exchange attraction describes the interaction between double paths and leads to a bound state in $d=2$. We thus expect the probability distribution for $\langle W^n(0,t)\rangle$ to have the same universal features as in the case of random signs.

We checked this by numerical simulations on the square lattice in Fig. 1, with phases θ independently chosen for each bond from a uniform distribution $-\pi<\theta<\pi$. Fluctuations in $\ln W(0,t)$ again grow as t^ω with $\omega=0.30\pm 0.05$, consistent with $\omega=\frac{1}{3}$. Another interesting quantity is the transverse fluctuations caused by the scattering. For each realization of randomness, using the weight $W(x,t)$, we calculate the expectations $[x]_{\rm av}$ and $[x^2]_{\rm av}$ which measure such fluctuations. These results are then averaged over many realizations. The results for $t\leq 4000$, averaged over 200 realizations, are plotted in Fig. 3. The transverse fluctuations $\langle[x]_{\rm av}^2\rangle$ and $\langle[x^2]_{\rm av}\rangle$ appear to converge to the same limit, although each is influenced by corrections to leading scaling. Their difference, also plotted in Fig. 3, grows approximately as t (a *subleading* power law). By considering both curves we have concluded that the leading asymptotic behavior is $t^{2\nu}$, with $\nu=0.68\pm 0.05$. We interpret these results as due to a distribution $W(x,t)$ with a width growing as $t^{1/2}$, while the center fluctuates as t^ν. Such fluctuations of the "center" can be visualized as strechings of a coarse-grained path length by x^2/t. The associated energy cost of streching is tolerated if sufficient free-energy fluctuations are available. As the free-energy fluctuations increase like t^ω, this implies the exponent identity $2\nu-1=\omega$. Using $\omega=\frac{1}{3}$, we obtain $\nu=\frac{2}{3}$ which is somewhat smaller than, though consistent

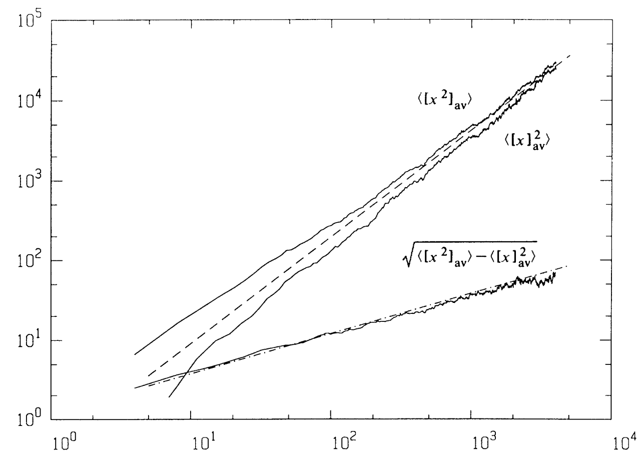

FIG. 3. Transverse fluctuations measured by $\langle[x^2]_{\rm av}\rangle$ and $\langle[x]_{\rm av}^2\rangle$ vs the path length t. The dashed line has slope $2\nu=\frac{4}{3}$. The difference between the two curves appears to grow linearly in t. The dashed-dotted line has slope $\frac{1}{2}$.

with, the numerical data in Fig. 3.

In conclusion, we have carried our numerical (transfer matrix) and analytical (replicas) studies of sums over directed paths with random signs or phases in $d=2$. Our results indicate that these sums have universal characteristics similar to the case of random magnitudes—the logarithm of the sum has a well-defined distribution with fluctuations that grow as $t^{1/3}$. The dominant contribution to the sums comes from paths that disperse as $t^{2/3}$ in the transverse direction. The replica analysis can be extended to higher dimensions: The replicated paths are always paired up (at $p=\frac{1}{2}$) and experience an exchange attraction. In $d=3$ there is still a bound state, and fluctuations should behave as directed polymers in a random 3D environment.[4] For $d>3$, although a bound state is still possible, unlike the directed-polymer[4] case the strength of the attraction (and hence the bound state) is not adjustable. We may speculate whether the unbinding of replica pairs at some $p<\frac{1}{2}$ in $d>3$ indicates the sign transition anticipated by NSS.[3] In earlier work on the hopping problem,[3,8] it was assumed that typical paths are diffusive with a width scaling as $t^{1/2}$. As the cigar-shaped area between two such paths scales as $t^{3/2}$, in a magnetic field B a typical phase change proportional to $Bt^{3/2}$ between paths was anticipated. We find that dominant paths are superdiffusive, implying a larger flux (proportional to $Bt^{1.7}$). Although the difference is small, since it effects a quantity that is potentially observable,[8] its relevance to future experiments cannot be ruled out. We are currently pursuing the question of fluctuations in conductivity in the presence of a magnetic field.

The research at MIT was supported by the NSF through Grant No. DMR-86-2038€. The authors acknowledge financial support from "Centro de Investigacion y Desarrollo" INTEVEP S.A. Venezuela (E.M.), and the Sloan Foundation (M.K.).

Note added.–In his Comment[9] (this issue) Zhang addresses very similar questions. On the basis of numerical results similar to $\langle [x]^2_{av} \rangle$ in Fig. 3, he concludes that $\nu=\frac{3}{4}$, and provides a replica justification for this result based on $\omega=\frac{1}{2}$. We disagree with his replica analysis and believe that the larger ν may be due to subleading corrections to scaling.

[1] S. F. Edwards and P. W. Anderson, J. Phys. F **5**, 965 (1975).

[2] For a recent review, consult K. Binder and A. P. Young, Rev. Mod. Phys. **58**, 801 (1986).

[3] V. L. Nguen, B. Z. Spivak, and B. I. Shklovskii, Pis'ma Zh. Eksp. Teor. Fiz. **41**, 35 (1985) [JETP Lett. **41**, 42 (1985)]; Zh. Eksp. Teor. Fiz. **89**, 11 (1985) [JETP Sov. Phys. **62**, 1021 (1985)].

[4] M. Kardar and Y.-C. Zhang, Phys Rev. Lett. **58**, 2087 (1987), and references therein.

[5] M. Kardar, Nucl. Phys. **B290 [FS20]**, 582 (1987).

[6] D. A. Huse, C. L. Henley, and D. S. Fisher, Phys. Rev. Lett. **55**, 2924 (1985).

[7] Y. Shapir and X.-R. Wang, Europhys. Lett. **4**, 10 (1987).

[8] U. Sivan, O. Entin-Wohlman, and Y. Imry, Phys. Rev. Lett. **60**, 1566 (1988), and references therein.

[9] Yi-Cheng Zhang, this issue, Phys. Rev. Lett. **62**, 979 (1989).