Order Parameter and Ginzburg-Landau Theory for the Fractional Quantum Hall Effect

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A new order parameter with a novel broken symmetry is proposed for the fractional quantum Hall effect, with the Laughlin state as the mean-field ground state. The classical Ginzburg-Landau theory of Girvin is derived microscopically from this starting point and exhibits all the phenomenology of the fractional quantum Hall effect.

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While there is now a good understanding of the properties of the states responsible for the fractional quantum Hall effect (FQHE) in the lowest Landau level,[1] a completely general characterization of these states has not yet been given. Girvin[2] has suggested that this might be done by invoking a superfluid analogy, in which the fluid is described by a complex scalar order parameter obeying a Ginzburg-Landau equation, and the vortex excitations are identified with the fractionally charged quasiparticles of Laughlin's theory.[1] In a later Letter, Girvin and MacDonald[2] (GM) showed that a certain modified density matrix exhibits algebraic off-diagonal long-range order in the Laughlin state, providing further evidence for the superfluid analogy.

In this Letter, I construct the superfluid analogy explicitly on a microscopic basis. An order parameter that shows genuine long-range order in the Laughlin state is constructed, related to, but distinct from, that of GM. The broken symmetry is identified, and the Ginzburg-Landau action is derived at the classical, linearized level. All the phenomenology of the FQHE at filling factors $\nu=1/q$ follows, and generalization to other filling factors can be made at least in principle. Physically, the order parameter describes the special correlations of the Laughlin state (binding of zeroes to particles).

We first exhibit a correlation function which possesses off-diagonal long-range order, indicating that the usual Laughlin state[3] is not a pure phase[4] and that a symmetry is broken. We use (i) a lowest-Landau-level-projected second-quantized field operator in the symmetric gauge,[5]

$$\psi(z)=\sum_{n=0}^{\infty} a_n u_n(z), \quad u_n=\frac{z^n e^{-|z|^2/4}}{(2\pi 2^n n!)^{1/2}}, \qquad (1)$$

where a_n is a destruction operator for the nth single-particle basis state u_n, and (ii) Laughlin's quasihole operator,[3] in first quantization,

$$U(z)=\prod_{i=1}^{N}(z_i-z), \qquad (2)$$

in the N-particle subspace. Note that while $\psi(z)$ removes a particle bodily from the fluid, leaving a hole of charge 1, $U(z)$ moves particles outwards by increasing their angular momentum about z, leaving a deficiency of charge $1/q$ there if the state is a *fluid state* of slowly varying density ρ close to $\rho_0=1/2\pi q$ with no positional long-range order. $U(z)^q$ thus leaves the same charge deficiency as $\psi(z)$, and the essence of the present approach is that these two types of hole states are physically equivalent, so that they have a nonzero overlap.[6]

In the normalized Laughlin ground state $|0_L;N\rangle$ for N particles, whose (unnormalized) coordinate representation is

$$\prod_{i<j}(z_i-z_j)^q \exp\left(-\tfrac{1}{4}\sum_i |z_i|^2\right),$$

we can show that

$$\langle 0_L;N|\tilde{U}^\dagger(z)^q\psi(z)\psi^\dagger(z')\tilde{U}(z')^q|0_L;N\rangle$$
$$=\rho_0^{-1}\langle 0_L;N+1|\rho(z)\rho(z')|0_L;N+1\rangle\rightarrow\rho_0 \qquad (3)$$

as $|z-z'|\rightarrow\infty$ with $|z|,|z'|>N$, which is not equal to

$$|\langle 0_L;N|\tilde{U}^\dagger(z)^q\psi(z)|0_L;N\rangle|^2$$

which vanishes identically. Here and below we denote

$$\tilde{U}(z)^q|\alpha\rangle=U(z)^q|\alpha\rangle/\langle\alpha||U(z)|^2|\alpha\rangle^{1/2},$$

where $|\alpha\rangle$ is a normalized fluid state.

Equation (3) shows that the Laughlin state is not a pure state.[4] A pure state, in which $\psi^\dagger U^q$ has a nonzero expectation value, can be constructed as

$$|0_L;\theta\rangle=\sum_{N=1}^{\infty}|\alpha_N|e^{-iN\theta}|0_L;N\rangle, \qquad (4)$$

where $\{|\alpha_N|^2\}$ is a binomial distribution function for N with mean $\bar{N}\gg 1$ and variance of order $\bar{N}$, and θ is arbitrary. For this state and arbitrary z,

$$\langle\Psi^\dagger(z)\rangle\equiv\langle\psi^\dagger(z)\tilde{U}(z)^q\rangle\rightarrow\rho_0^{1/2}e^{i\theta}$$

as $\bar{N}\rightarrow\infty$, and this defines our order parameter. From now on, fluid states $|\alpha\rangle$ will be taken to be pure states with nonzero order parameter. Physical properties will be more transparent when working with pure states.

Since $\psi^\dagger$ increases N, the particle number, by 1, and U^q increases $M(z)$, the angular momentum about z, by qN, $\psi^\dagger U^q$ breaks the symmetry generated by $\frac{1}{2}N+M/qN$, while $\frac{1}{2}N-M/qN$ is unbroken. Ψ characterizes the Laughlin state, since

$$|0_L;N\rangle=\left(\int d^2z\,\psi^\dagger(z)U(z)^q e^{-|z|^2/4}\right)^N|0\rangle$$

up to a normalization factor, in exact analogy with the ground state of a Bose gas or BCS superconductor. Thus the Laughlin state as in (4) is precisely the mean-field theory of the FQHE.

Note that $\langle\Psi^\dagger(z)\rangle$ is a *local* order parameter, even though $U(z)$ acts on particles far from z, because an (in principle distinct) value can be associated with each point z; this allows it to have thermodynamic significance in a Ginzburg-Landau description, as will be shown.

While the present order parameter resembles that of GM in involving a particle bound to a flux tube (here U^q), it differs in that we find true long-range order in the Laughlin state whereas GM find only algebraic order. The algebraic order of GM is apparently an artifact of their choice of flux operator. We note that any choice of flux operator in place of U^q gives a candidate order parameter for some fluid ground state of filling factor $1/q$, since by a Berry-phase calculation[7] the flux operator will be fermionic if q is odd, and the counting of charge makes the combination, like $\psi^\dagger U^q$, a locally neutral Bose-type operator, which may Bose condense, giving a liquid state.

In constructing the Ginzburg-Landau action, we will use states

$$|\alpha;z,n\rangle=(2^n n!)^{-1/2}(2\,\partial/\partial z+i\mathcal{A}_-)^n\tilde{U}(z)^q|\alpha\rangle\,, \quad (5)$$

where $|\alpha\rangle$ is a (pure) fluid state. Equation (5) is the generalization to the normalized hole state $\tilde{U}(z')^q|\alpha\rangle$ of the expansion of the unnormalized state $U(z')^q|\alpha\rangle$ in powers of $z'-z$ about some point z. The vector potential $\mathcal{A}_-=\mathcal{A}_x-i\mathcal{A}_y$ accounts for the normalization and generalizes[8] analyticity $\partial/\partial\bar{z}\equiv 0$,

$$(2\,\partial/\partial\bar{z}+i\mathcal{A}_+)\tilde{U}(z)^q|\alpha\rangle\equiv 0\,,$$

giving us

$$i\mathcal{A}_-(z)=q\int\frac{d^2z'}{z'-z}\langle\tilde{U}^\dagger(z)^q\rho(z')\tilde{U}(z)^q\rangle\,, \quad (6)$$

where $\rho(z')=\psi^\dagger(z')\psi(z')$ is the density in the lowest Landau level. Equation (6) can equivalently be obtained from adiabatic transport of the hole.[7] $\mathcal{A}_-(z)$ can be calculated approximately by first (exactly) commuting $\rho(z')$ to the right to give

$$i\mathcal{A}_-(z)=q\int\frac{d^2z'}{z'-z}\,\frac{\langle\alpha|U^\dagger(z)^qU(z)^qR_q(z',z)|\alpha\rangle}{\langle\alpha|U^\dagger(z)^qU(z)^q|\alpha\rangle}\,, \quad (7)$$

where $R_n(z',z)$ is a one-body operator. Equation (7) is now approximated by the insertion of $|\alpha\rangle\langle\alpha|$ between U^q and R_q, in which case, remarkably,

$$\tfrac{1}{2}\epsilon_{\alpha\beta}\partial_\alpha\mathcal{A}_\beta\equiv i\,\partial\mathcal{A}_-/\partial\bar{z}=-\pi q\langle\rho(z)\rangle\,, \quad (8)$$

where $\alpha=x,y$, $\partial_x=\partial/\partial x$, etc. For a circular droplet of density $\langle\rho\rangle=\rho_0=(2\pi q)^{-1}$, we find $\mathcal{A}_-=\frac{1}{2}i\bar{z}$, the symmetric gauge.

Because of the existence of the order parameter, we can relate hole states by the approximate expansion

$$\psi(z)|\alpha\rangle=\sum_{m=0}^{\infty}\beta_m|\alpha;z,m\rangle\,,$$
$$\beta_m=(2^m m!)^{-1/2}(2\,\partial/\partial\bar{z}+ia_+)^m\langle\Psi(z)\rangle\,, \quad (9)$$

where $\mathbf{a}=\mathbf{A}-\boldsymbol{\mathcal{A}}$ satisfies[2]

$$\epsilon_{\alpha\beta}\partial_\alpha a_\beta=2\pi q(\langle\rho\rangle-\rho_0)\,. \quad (10)$$

In Eq. (9) the $|\alpha;z,m\rangle$ for different m were treated as orthonormal; if $|\alpha\rangle$ is the ground state, this is correct; otherwise, there are overlaps between different n values because $\epsilon_{\alpha\beta}\partial_\alpha\mathcal{A}_\beta$ is not constant. These overlaps and the norms of the states may be calculated recursively; the definition of $\mathcal{A}_-$ implies that $|\alpha;z,0\rangle$ is normalized and orthogonal to $|\alpha;z,1\rangle$ in general. An orthonormal set of states can be constructed by the Gram-Schmidt method. This introduces corrections to the $|\alpha;z,n\rangle$ which, however, can be neglected in the *linearized* calculation described below, because a derivative of a slowly varying expectation value of either ρ or Ψ always appears, which is certainly already of first order in the deviation from the ground-state value. Hence, we may use (5) in (9). We see that

$$\langle\rho\rangle\simeq|\langle\Psi\rangle|^2+\cdots\,.$$

We now derive the Ginzburg-Landau theory for $\langle\Psi\rangle$ by first obtaining approximate equations of motion for $(id/dt)\langle\Psi^\dagger\rangle$ and then writing down an action whose variation gives these equations.

The Hamiltonian projected into the lowest Landau level contains only potential-energy terms:

$$H=-\int d^2z\,V_{\rm ext}(z)\rho(z)+\tfrac{1}{2}\int d^2z_1d^2z_2V(z_1-z_2)\psi^\dagger(z_1)\psi^\dagger(z_2)\psi(z_2)\psi(z_1)\,, \quad (11)$$

where $V(z)$ is a function of $|z|$ only.

Working in the Heisenberg picture, we straightforwardly obtain

$$\left\langle i\frac{d\psi^\dagger(z)}{dt}\tilde{U}(z)^q\right\rangle=\sum_{n=0}^{\infty}(2\pi n!)^{-1}\left[\frac{\partial^n}{\partial z^n}\int d^2z'\,V_{\rm ext}(z')e^{-|z'-z|^2/2}\right]\left[2\frac{\partial}{\partial\bar{z}}-ia_+\right]^n\langle\Psi^\dagger(z)\rangle$$
$$-V_H(0)\langle\Psi^\dagger(z)\rangle+\nabla^2V_H(0)\left[\frac{\partial}{\partial\bar{z}}-\tfrac{1}{2}ia_+\right]\left[\frac{\partial}{\partial z}-\tfrac{1}{2}ia_-\right]\langle\Psi^\dagger(z)\rangle. \quad (12)$$

In the interparticle potential terms, use has been made of (9), linearized in the deviation of $\langle\Psi^\dagger\rangle$ from its ground-state value, and higher derivatives have been dropped. The "Hartree-type potential,"

$$V_H(z_1-z)=\int d^2z_2\,V(z_1-z_2)\langle\tilde{U}^\dagger(z)^q\rho(z_2)\tilde{U}(z)^q\rangle\,, \quad (13)$$

is evaluated in the ground state and is then a function of $|z_1-z|$ only. I have neglected in (12) terms arising from taking $\langle\Psi^\dagger\rangle$ to be its ground-state value, and keeping the change in V_H to linear order; these "exchangelike" terms might give mass, quartic interaction, or additional gradient-squared terms in the Ginzburg-Landau long-wavelength action. The omission of these terms, which has no effect on the physics derived here, is our main dynamical approximation.

The remainder of the equation of motion is obeyed by the projection of $(i\,d/dt)\tilde{U}(z)^q|\alpha\rangle$ onto the basis set (5); one finds

$$i\frac{d}{dt}\tilde{U}(z)^q|\alpha\rangle=\Phi(z)|\alpha;z,0\rangle+\sum_{n=1}^{\infty}(2^n n!)^{-1/2}\left[\left[2\frac{\partial}{\partial\bar{z}}\right]^n\Phi_C(z)\right]|\alpha;z,n\rangle\,, \quad (14)$$

$$\Phi_C(z)=\frac{\langle U^\dagger(z)^q(i\,d/dt)U(z)^q\rangle}{\langle U^\dagger(z)^qU(z)^q\rangle}\simeq\int d^2z'\,2\frac{\partial V_{\rm ext}}{\partial\bar{z}'}(z')\sum_{r=0}^{q-1}\frac{\langle R_{r+1}(z',z)\rangle}{z'-z}$$
$$-\int d^2z_1\,d^2z_2\,2\frac{\partial V}{\partial\bar{z}_1}(z_1-z_2)\sum_{r=0}^{q-1}\frac{\langle\psi^\dagger(z_2)R_{r+1}(z_1,z)\psi(z_2)\rangle}{z_1-z}\,, \quad (15)$$

and $\Phi=\mathrm{Re}\Phi_C$; I have approximated by decoupling as in (7) and (8) and also dropped a term in the two-body part that involves both $[\rho(z_2),U(z)^q]$ and R_{r+1}, which is a higher-order correlation. Then, by manipulations similar to those used in (8) and (12), we obtain

$$2\frac{\partial\Phi_C}{\partial\bar{z}}=-\int\frac{d^2z'}{2\pi q}\,2\frac{\partial V_{\rm ext}}{\partial\bar{z}'}(z')\sum_{r=0}^{q-1}\frac{|z'-z|^{2r}e^{-|z-z'|^2/2}}{2^r r!}-2\pi q\nabla^2V_H(0)\langle\Psi^\dagger(z)\rangle\left[\frac{\partial}{\partial\bar{z}}+\tfrac{1}{2}ia_+\right]\langle\Psi(z)\rangle\,, \quad (16)$$

up to higher gradients of $\langle\Psi\rangle$. From the exact expressions for a,Φ,

$$da_+/dt+2\,\partial\Phi/\partial\bar{z}\equiv2\,\partial\Phi_C/\partial\bar{z}\,, \quad (17)$$

and the right-hand side of (16) can be interpreted as the drift-motion current due to the external and interparticle potentials.

Finally,

$$i\frac{d}{dt}\langle\Psi^\dagger(z)\rangle=\sum_{n=0}^{\infty}(2\pi n!)^{-1}\left[\frac{\partial^n}{\partial z^n}\int d^2z'\,V_{\rm ext}(z')e^{-|z-z'|^2/2}\right]\left[2\frac{\partial}{\partial\bar{z}}-ia_+\right]^n\langle\Psi^\dagger(z)\rangle$$
$$-\sum_{n=1}^{\infty}(2\pi qn!)^{-1}\left[\frac{\partial^{n-1}}{\partial\bar{z}^{n-1}}\int d^2z'\frac{\partial V_{\rm ext}}{\partial\bar{z}'}(z')\sum_{r=0}^{q-1}\frac{|z'-z|^{2r}}{2^r r!}e^{-|z-z'|^2/2}\right]\left[2\frac{\partial}{\partial z}-ia_-\right]^n\langle\Psi^\dagger(z)\rangle$$
$$+[\Phi(z)-V_H(0)]\langle\Psi^\dagger(z)\rangle+\nabla^2V_H(0)\left[\frac{\partial}{\partial\bar{z}}-\tfrac{1}{2}ia_+\right]\left[\frac{\partial}{\partial z}-\tfrac{1}{2}ia_-\right]\langle\Psi^\dagger(z)\rangle \quad (18)$$

to linear order in deviations from the ground state. Note the similar structure of the first two terms.

Since we are working at indefinite particle number and area, we must add chemical potential and pressure terms to the Hamiltonian, which can be incorporated in $V_{\rm ext}$; this takes the form of a constant potential in the interior of the droplet, with slowly rising confining walls near the edge, and can be arranged to cancel in the ground-state case the terms on the right-hand side of (18) with no gradients of $\langle\Psi^\dagger\rangle$. Thus $(i\,d/dt)\langle\Psi^\dagger\rangle$ vanishes in the interior of the droplet in the ground-state case where $\langle\Psi^\dagger\rangle$ is uniform in space.

With the remainder of $V_{\rm ext}$ omitted for clarity, Eqs. (10) and (16)–(18) can be obtained by variation of the Lagrangian density

$$L=\Psi^\dagger i\frac{d}{dt}\Psi-\tfrac{1}{2}C\left[\left(2\frac{\partial}{\partial z}-ia_-\right)\Psi^\dagger\right]\left(2\frac{\partial}{\partial \bar z}+ia_+\right)\Psi+\Phi[|\Psi|^2-\rho_0-(2\pi q)^{-1}\epsilon_{\alpha\beta}\partial_\alpha a_\beta]-\frac{1}{4\pi q}\epsilon_{\alpha\beta}a_\alpha\frac{d}{dt}a_\beta, \qquad (19)$$

where $C=\frac{1}{2}\nabla^2V_H(0)$ (as usual, L is determined by the equations of motion only up to total time derivatives).

The same linearized equations of motion can be solved for plane-wave excitations to yield a dispersion relation

$$\omega^2=C^2+\tfrac{1}{2}Ck^2(C+\tfrac{1}{2}Ck^2);$$

thus this collective mode has a gap $\omega=C$ as $k\to0$ which is due to the long-range gauge forces in the action (19) (the Anderson-Higgs mechanism). The roton minimum[9] at a larger wave vector $k\sim\rho_0^{1/2}$ is not obtained within the present approximation. The fractional statistics[7,10] of the vortices and the quantized Hall conductance also follow from (19).

The physical meaning of the order parameter $\langle\Psi^\dagger(z)\rangle$ is that it is the amplitude for finding a particle at z at the zeroes of the many-particle wave function, and by (9) its gradients represent the amplitudes for displacements from the zeroes. A nonzero displacement leads to a higher Hartree energy (13) (which just involves the two-particle correlation function) and hence to the stiffness constant $C=\frac{1}{2}\nabla^2V_H(0)$. The long-range gauge forces are related to those in Laughlin's plasma analogy,[3] but here take on a dynamic as well as static role.

It should be possible to describe quantum fluctuations about the Laughlin state by quantizing the action (19), but this should be done with care to ensure a connection with the microscopic description. Since the Laughlin state is exact for the pseudopotential Hamiltonian[11] with $V_1,V_3,\ldots,V_{q-2}\neq0$, $n\geq q$, quantum fluctuations will be controlled by the size of V_n, $n\geq q$.

A similar order parameter can be constructed for general filling factors $\nu=p/q$ with use of $\psi^{\dagger p}U^q$, and also extended to spin-singlet states;[12] these extensions and details of the present work will be given elsewhere.[13]

A brief report of part of this work was given previously.[14] After this work was completed, we received a preprint from Rezayi and Haldane,[15] who studied related order parameters numerically, and showed that they are nonzero in FQHE states at $\nu=\frac{1}{3},\frac{2}{5}$ but vanish in compressible states. We also learned of work by Zhang, Hansson, and Kivelson[16] on the Ginzburg-Landau action.

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[5]The conventions here are that **B** points in the negative $\hat{\mathbf{z}}$ direction, coordinates in the plane are labeled by $z=x+iy$, $\partial/\partial z\equiv\frac{1}{2}(\partial/\partial x-i\,\partial/\partial y)$, the magnetic length $l=\hbar=1$, and the magnitude but not the sign of the electric charge is absorbed into the potentials. All results are gauge covariant.

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