Electromagnetic Properties of Generalized Majorana Particles

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We prove a theorem stating that a massive Majorana particle (a *CPT*-self-conjugate particle) with *arbitrary spin J* can possess only an anapole moment and multipoles of that. We also show that massless Majorana particles, except those of spin $\frac{1}{2}$, do not have any single-photon electromagnetic form factor.

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Soon after the extensive studies of the electromagnetic properties of the spin-$\frac{1}{2}$ Majorana particle[1] which, in particular, showed that these particles possess only an anapole moment[2] (which is different from the charge, magnetic, and electric dipole moments), Radescu[3] extended this result to Majorana fermions with higher spin. Defining a Majorana particle to be a *CPT*-self-conjugate particle, it is worth inquiring whether the property that a Majorana can only have an anapole moment (and higher multipoles of it) also holds for Majorana bosons. A preliminary investigation pertaining to the spin-1 boson[4] has found that this is indeed the case. In this Letter we present a general proof of the above property. We also demonstrate that massless Majorana particles, with the exception of those with spin $\frac{1}{2}$, *have no single-photon electromagnetic couplings at all.* We will use the formalism of the multipole expansion of the electromagnetic current. An alternative derivation which is manifestly covariant will be presented elsewhere.[5]

A massive particle with spin J can be specified by the state $|\mathbf{p};J,M\rangle$ with $\mathbf{p}$ its momentum and M the spin projection along a given axis, say $\hat{\mathbf{z}}$ in its rest frame. We will denote the *CPT* operation by Θ. For a Majorana particle we have

$$\Theta|\mathbf{p};J,M\rangle=\eta_J(M)|\mathbf{p};J,-M\rangle , \tag{1}$$

where $\eta_J(M)$ is a phase factor satisfying

$$\eta_J(M')=(-1)^{M'-M}\eta_J(M) , \tag{2}$$

as can be deduced by using the lowering and raising operators of the angular momentum combined with *CPT*.

Let us now consider the electromagnetic transition process $M(p_i)\to M(p_f)\gamma(k)$ with $M(p)$ being a *CPT*-self-conjugate particle of momentum p and $\gamma(k)$ a virtual photon with momentum $k=p_i-p_f$. To derive the electrostatic form factors, the multipole expansion analysis is conducted in the Breit frame in which

$$p_i=(q_0,\mathbf{q}),\quad p_f=(q_0,-\mathbf{q}),\quad k=(0,2\mathbf{q}) . \tag{3}$$

Here, we look at the matrix elements of the electromagnetic current $j_\mu(0)$ between the initial state $|\mathbf{q};J,M_i\rangle$ and the final state $|-\mathbf{q};J,M_f\rangle$, i.e.,

$$\begin{aligned}&\langle -\mathbf{q};J,M_f|j_0(0)|\mathbf{q};J,M_i\rangle=\Pi^J_{M_fM_i}(\mathbf{q}) ,\\ &\langle -\mathbf{q};J,M_f|\mathbf{j}(0)|\mathbf{q};J,M_i\rangle=\boldsymbol{\Pi}^J_{M_fM_i}(\mathbf{q}) .\end{aligned} \tag{4}$$

It follows from the current conservation that the spatial part of the matrix element should be transverse with respect to $\mathbf{q}$: $\mathbf{q}\cdot\boldsymbol{\Pi}^J_{M_fM_i}(\mathbf{q})=0$. Taking this constraint into account, the multipole expansion of the matrix elements of the electromagnetic current is given by

$$\begin{aligned}\Pi^J_{M_fM_i}(\mathbf{q})&=\sum_{L=0}^{2J}(-1)^{J-M_i}\begin{pmatrix}J & L & J\\ -M_f & -M & M_i\end{pmatrix}(i|\mathbf{q}|)^L Q_{JL}(\mathbf{q}^2)Y_{LM}(\hat{\mathbf{q}}) ,\\ \boldsymbol{\Pi}^J_{M_fM_i}(\mathbf{q})&=\sum_{\substack{L=1\\ \lambda=0,1}}^{2J}(-1)^{J-M_i}\begin{pmatrix}J & L & J\\ -M_f & -M & M_i\end{pmatrix}(i|\mathbf{q}|)^{L+\lambda}Q^{(\lambda)}_{JL}(\mathbf{q}^2)\mathbf{Y}^{(\lambda)}_{LM}(\hat{\mathbf{q}}) ,\end{aligned} \tag{5}$$

where $M=M_i-M_f$,

$$\begin{pmatrix}J & L & J\\ -M_f & -M & M_i\end{pmatrix}$$

is the Wigner $3j$ symbol, $Y_{LM}(\hat{\mathbf{q}})$ are the ordinary spherical harmonics, and $\mathbf{Y}^{(\lambda)}_{LM}(\hat{\mathbf{q}})$ are the transverse vector spherical harmonics defined by

$$\begin{aligned}\mathbf{Y}^{(0)}_{LM}(\hat{\mathbf{q}})&=\frac{1}{[L(L+1)]^{1/2}}\mathbf{q}\times\nabla Y_{LM}(\hat{\mathbf{q}}) ,\\ \mathbf{Y}^{(1)}_{LM}(\hat{\mathbf{q}})&=\hat{\mathbf{q}}\times\mathbf{Y}^{(0)}_{LM}(\hat{\mathbf{q}}) .\end{aligned} \tag{6}$$

The expansions (5) were derived in Ref. 6 using the Wigner-Eckart theorem. It follows from the hermiticity of J_μ that all the form factors Q_{JL} and $Q_{JL}^{(\lambda)}$ are real.

Theorem 1.— If the underlying theory is *CPT* invariant and the initial and final particles are *CPT* self-conjugate, then

$$Q_{JL}(\mathbf{q}^2)=Q_{JL}^{(0)}(\mathbf{q}^2)=0\,, \tag{7}$$

i.e., only one set of multipole moments, $Q_{JL}^{(1)}(\mathbf{q}^2)$, exists.

Proof.— Defining $|\psi_\Theta\rangle=\Theta|\psi\rangle$, with O being an operator in the Hilbert space of states $|\psi\rangle$, we have

$$\langle\psi|O|\psi'\rangle=\langle\psi'_\Theta|\Theta O^\dagger\Theta^{-1}|\psi_\Theta\rangle\,. \tag{8}$$

Using this formula and the phase relation (2), we obtain

$$\langle -\mathbf{q};J,M_f|j_\mu(0)|\mathbf{q};J,M_i\rangle = (-1)^{M_f-M_i+1}\langle \mathbf{q};J,-M_i|j_\mu(0)|-\mathbf{q};J,M_f\rangle\,. \tag{9}$$

The additional minus sign is because the current operator $j_\mu(0)$ is odd under a *CPT* transformation. It follows from the expansion (5) and the symmetry property of the Wigner $3j$ symbol,[7]

$$\begin{pmatrix} J & L & J \\ M_i & -M & -M_f \end{pmatrix}=(-1)^{2J+L}\begin{pmatrix} J & L & J \\ -M_f & -M & M_i \end{pmatrix}, \tag{10}$$

that

$$Q_{JL}(\mathbf{q}^2)=-Q_{JL}(\mathbf{q}^2)\,,\qquad Q_{JL}^{(\lambda)}(\mathbf{q}^2)=(-1)^{\lambda+1}Q_{JL}^{(\lambda)}(\mathbf{q}^2)\,, \tag{11}$$

for any J. Therefore $Q_{JL}(\mathbf{q}^2)=Q_{JL}^{(0)}(\mathbf{q}^2)=0$ and the theorem is proved. The matrix elements of the current in the Breit frame can be written as

$$\langle -\mathbf{q};J,M_f|j_0(0)|\mathbf{q};J,M_i\rangle=0\,,$$
$$\langle -\mathbf{q};J,M_f|\mathbf{j}(0)|\mathbf{q};J,M_i\rangle=\sum_{L=1}^{2J}(-1)^{J-M_i}\begin{pmatrix} J & L & J \\ -M_f & -M & M_i \end{pmatrix}Q_{JL}^{(1)}(\mathbf{q}^2)(i|\mathbf{q}|)^{L+1}\mathbf{Y}_{LM}^{(1)}(\hat{\mathbf{q}})\,. \tag{12}$$

The matrix elements of the current in an arbitrary frame can be obtained by an appropriate boost. The theorem is valid not only for point particles but also for extended objects in the nonrelativistic limit since the proof relies only on the rotation behavior and *CPT* transformation of the state.

The multipole moments characterized by $\mathbf{Y}_{LM}^{(1)}(\hat{\mathbf{q}})$ are the toroidal moments.[2,6] The total number of these toroidal moments could be at most $2J$ for a particle of spin J. As an example, the neutral pion has no electromagnetic structure, the spin-$\frac{1}{2}$ Majorana neutrino can possess only one electromagnetic form factor, and the Z boson can possess up to two form factors. Furthermore, *if C is conserved then all form factors vanish.* This fact shows that the toroidal moments are C-violating quantities in contrast to the magnetic and electric dipole moments. In addition, the anapole moment has the special feature that it interacts only with the external current, in contrast to the magnetic and electric dipole moments which interact with the external fields. On the other hand, if P is conserved and C and T are violated (*CPT* conserved), then the toroidal moments $Q_{JL}^{(1)}(\mathbf{q}^2)$ vanish for odd values of L, and if T is conserved and C and P are violated, then the $Q_{JL}^{(1)}(\mathbf{q}^2)$ for even values of L vanish. This has been shown explicitly for the case of the Z boson[4] which at the one-loop order (*CP* is conserved) possesses only *one* anapole moment.

Now we turn to the massless case. The state $|\mathbf{q},\lambda\rangle$ of a massless particle of spin J is labeled by the momentum $\mathbf{q}$ and the helicity λ which can only assume two values, $\pm J$. For a massless Majorana particle, the analog of Eq. (1) is

$$\Theta|\mathbf{q};\lambda\rangle=\eta(\mathbf{q},\lambda)|\mathbf{q},-\lambda\rangle\,, \tag{13}$$

where

$$\eta^*(\mathbf{q},-\lambda)\eta(\mathbf{q},\lambda)=(-1)^{2J}\,, \tag{14}$$

as a consequence of $\Theta^2=(-1)^{2J}$.

Theorem 2.— If the underlying theory is *CPT* invariant, and the initial and the final particle states are *CPT* self-conjugate and massless, the matrix elements of the electromagnetic current between the initial and final states, $\langle -\mathbf{q};\lambda_f|j_\mu(0)|\mathbf{q};\lambda_i\rangle$, vanish for all spins except $J=\frac{1}{2}$. The matrix elements of the spin-$\frac{1}{2}$ particles contain only one form factor given by the anapole moment.

Proof.— Choose a coordinate system such that $\mathbf{q}$ is parallel to $\hat{\mathbf{z}}$ and define

$$Q_{\lambda_f\lambda_i}^J(\mathbf{q})=\langle -\mathbf{q};\lambda_f|j_0(0)|\mathbf{q};\lambda_i\rangle\,, \tag{15}$$

$$Q_{\lambda_f\lambda_i}^{J\pm}(\mathbf{q})=\langle -\mathbf{q};\lambda_f|j_\pm(0)|\mathbf{q};\lambda_i\rangle\,, \tag{16}$$

with $j_\pm(0)=j_x(0)\pm ij_y(0)$. First, we consider $Q_{\lambda_f\lambda_i}^J(\mathbf{q})$. Inserting a rotation about $\hat{\mathbf{z}}$ through ϕ [this technique was used by Weinberg and Witten in Ref. 8; they arrived at the same conclusion for $\langle -\mathbf{q};\lambda|j_\mu(0)|\mathbf{q};\lambda\rangle$], we have

$$\langle -\mathbf{q};\lambda_f|j_0(0)|\mathbf{q};\lambda_i\rangle=\exp[i\phi(\lambda_i+\lambda_f)]\times\langle -\mathbf{q};\lambda_f|j_0(0)|\mathbf{q};\lambda_i\rangle\,. \tag{17}$$

Then $Q_{\lambda_f\lambda_i}^J(\mathbf{q})$ could be nonzero only for $\lambda_f=-\lambda_i=\lambda$. To apply a *CPT* operation, we relate the state $|-\mathbf{q};\lambda\rangle$ to the state $|\mathbf{q};\lambda\rangle$ through a π rotation about the y axis:

$$|-\mathbf{q};\lambda\rangle=c(\mathbf{q};\lambda)\exp(-i\pi J_y)|\mathbf{q};\lambda\rangle\,, \tag{18}$$

with $c(\mathbf{q};\lambda)$ being an arbitrary phase factor. Using Eqs.

(8), (13), (14), and (18) we obtain

$$Q^{J}_{\lambda\,-\lambda}(\mathbf{q}) = -Q^{J}_{\lambda\,-\lambda}(\mathbf{q}) , \tag{19}$$

for any J. Therefore, $Q^{J}_{\lambda\,-\lambda}(\mathbf{q})=0$. Similarly, the symmetry with respect to rotations about $\hat{\mathbf{z}}$ requires $\lambda_i+\lambda_f=-1$ for $Q^{J+}_{\lambda_f\lambda_i}(\mathbf{q})$ and $\lambda_i+\lambda_f=+1$ for $Q^{J-}_{\lambda_f\lambda_i}(\mathbf{q})$ and this can be satisfied only for $J=\frac{1}{2}$. The two nonvanishing matrix elements

$$Q_{\pm}=\langle -\mathbf{q};\mp\tfrac{1}{2}\,|\,j_{\pm}(0)\,|\,\mathbf{q};\mp\tfrac{1}{2}\rangle$$

may be related through a combined action of *CPT* and the π rotation about the y axis as in the case of $j_0(0)$. The relation is

$$Q_{+}(\mathbf{q}) = -c^{*}(\mathbf{q};-\tfrac{1}{2})c(\mathbf{q};\tfrac{1}{2})Q_{-}(\mathbf{q}) , \tag{20}$$

with $c(\mathbf{q};\pm\frac{1}{2})$ being the phase factors in (18) and only one dynamical form factor is left over. Theorem 2 is proved. This rules out the possibility of the graviton and the massless gravitino having a single-photon electromagnetic form factor.

Similar theorems may also be proved for the matrix elements of photoproduction

$$\langle \mathbf{q};J,M_1;-\mathbf{q};J,M_2\,|\,j_\mu(0)\,|\,0\rangle$$

by only using spin and statistics[5] (i.e., the fact that the final Majorana particles are identical particles).

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