Effective-Field-Theory Model for the Fractional Quantum Hall Effect

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Starting directly from the microscopic Hamiltonian, we derive a field-theory model for the fractional quantum Hall effect. By considering an approximate coarse-grained version of the same model, we construct a Landau-Ginzburg theory similar to that of Girvin. The partition function of the model exhibits cusps as a function of density and the Hall conductance is quantized at filling factors $\nu=(2k-1)^{-1}$ with k an arbitrary integer. At these fractions the ground state is incompressible, and the quasiparticles and quasiholes have fractional charge and obey fractional statistics. Finally, we show that the collective density fluctuations are massive.

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Despite the successes of the microscopic theories[1-3] of the fractional quantum Hall effect (FQHE),[4] it is still important to develop an effective-field-theory model analogous to the Landau-Ginzburg theory of superconductivity. An important step in this direction was taken by Girvin[5] and by Girvin and MacDonald,[6] who proposed a field-theory model, containing a complex scalar field ϕ coupled to a vector field $(a_0,\mathbf{a})$ with a Chern-Simons action (or topological mass term). This model exhibits vortex solutions with finite energy and fractional charge which can be identified with Laughlin's quasiparticles and quasiholes. The amplitude fluctuations of the ϕ field are massive and are identified with the density-fluctuation modes of the single mode approximation.[7,8] There is, however, no explanation for why the Hall conductance is quantized at certain specific fractional values, and in Ref. 6 it is also argued that the phase-fluctuation modes remain massless, contrary to the belief (based on the microscopic models) that *all* elementary excitations above the ground state have a finite gap, corresponding to an incompressible quantum liquid. Despite this, the model in Refs. 5 and 6 provides an important step towards a complete macroscopic description.

In this Letter, we derive a related model directly from the microscopic Hamiltonian, and demonstrate that it explains almost all known aspects of the FQHE. The coefficient of the Chern-Simons term in our case is determined by demanding that the elementary quanta of the ϕ field obey Fermi statistics, and the model exhibits cusps in the partition function at densities $n=n_B/(2k-1)$ (where n_B is the density of states in the lowest Landau level) corresponding to uniform solutions. For densities near an odd-integer filling fraction, the homogeneous state has an energy $\sim B|\delta n|$. In fact, the lowest-lying charged excitations involve a nonuniform charge density; they are spatially localized vortices with the same charge and statistics as the quasiparticles in Laughlin's approach.[1,9,10] We also show that the amplitude fluctuation of the ϕ field has a gap and can be identified with the collective density fluctuations[7,8] while the phase fluctuation of the ϕ field, or the Goldstone boson, is "eaten" by the vector field $(a_0,\mathbf{a})$ as a result of the Anderson-Higgs mechanism and disappears entirely from the spectrum.

We start from the following second-quantized many-body Hamiltonian

$$H=\int d^2r\,\phi^*(\mathbf{r})\left[\frac{1}{2m}[-i\nabla-e\mathbf{A}(\mathbf{r})-e\mathbf{a}(\mathbf{r})]^2+eA^0(\mathbf{r})\right]\phi(\mathbf{r})+\tfrac{1}{2}\int d^2r\,d^2r'\,\phi^*(\mathbf{r})\phi^*(\mathbf{r}')V(\mathbf{r}-\mathbf{r}')\phi(\mathbf{r})\phi(\mathbf{r}')\,, \qquad (1)$$

where

$$a^i(\mathbf{r})=\frac{\theta}{\pi e}\epsilon^{ij}\int d^2r'\frac{r_j-r_j'}{|\mathbf{r}-\mathbf{r}'|^2}\phi^*(\mathbf{r}')\phi(\mathbf{r}')\,, \qquad (2)$$

and where we have set $\hbar=c=1$. This Hamiltonian describes a system of identical particles with mass m which are created by the complex field operator ϕ. These particles interact via a two-dimensional gauge potential $\mathbf{a}$, and a two-body potential V. They are also coupled to an external electromagnetic field A^μ. From (2) it is clear that $\mathbf{a}$ is nothing but the "statistical" gauge potential employed in Refs. 11 and 12. Thus, if we take the ϕ field to be bosonic, the above Hamiltonian describes "anyons" obeying θ statistics (i.e., the wave function changes by a phase θ under the interchange of particles). For $\theta=(2k-1)\pi$ with k an arbitrary integer, this is simply the Hamiltonian for spin-polarized electrons in an external electromagnetic field interacting via the two-body potential $V(\mathbf{r})$.

From (2) we immediately get the following expression for the "statistical" gauge field b

$$b(\mathbf{r}) = -\epsilon^{ij}\partial_i a_j(\mathbf{r}) = \frac{2\theta}{e}|\phi(\mathbf{r})|^2 \equiv s|\phi(\mathbf{r})|^2 \left(\frac{2\pi}{e}\right). \tag{3}$$

which corresponds to associating $\theta/\pi = s$ units of flux to each particle. Changing between different k's corresponds to singular gauge transformations.

We can incorporate the constraint (3) by means of a Lagrange multiplier field a_0, and we add a chemical potential μ, which leads to the following coherent-state path-integral representation for the partition function:

$$Z[A^\mu] = \int [d\phi][da_i^T][da_0] e^{iS[\phi,a_i^T,a_0]}, \tag{4}$$

where a_i^T is a transverse gauge field (i.e., satisfying $\partial^i a_i^T = 0$), and $S = \int dt\, d\mathbf{r}\, \mathcal{L}$ with

$$\mathcal{L} = i\phi^* \partial_0 \phi - H(\phi) + \mu\phi^*\phi - a_0[(e^2/2\theta)\epsilon^{ij}\partial_i a_j^T + e\phi^*\phi]. \tag{5}$$

The term $\sim a^0$ in this expression is nothing but the Chern-Simons action in the radiation gauge.[13-15] The usual form of the Chern-Simons term can be obtained by reintroducing the (infinite) gauge volume (i.e., by reversing the usual Faddeev-Popov gauge fixing procedure[16]). We want to emphasize this rather interesting result. Several authors have pointed out that the excitations of two-dimensional field theories with Chern-Simon terms exhibit fractional statistics, and our derivation starting from the anyon formulation of Wilczek clearly demonstrates this. Also note that the size of the topological mass term is the one obtained both in our previous analysis of topologically massive (2+1)-dimensional QED,[17] and in the work of Semenoff.[18,19] So far we have made no approximations, other than those involving the intrinsic ambiguities of the coherent-state path integral itself.

In order to apply mean-field theory, we must first integrate out the short-distance fluctuations of the $\phi(\mathbf{r})$ field to obtain an effective action which describes the physics at distance scales larger than the magnetic length. We are currently engaged in carrying out such a calculation. For now, we make a simple *Ansatz* which we think is valid in the quantum Hall regime: The effective action is of the same form as the microscopic action, but with a renormalized stiffness constant κ replacing the bare mass $1/m$, and an effective interaction strength λ replacing the nonlocal interaction $V(r)$. Since the Chern-Simons term embodies the statistics of the bare particles, we do not expect it to renormalize. Higher-order derivative terms in a_μ, such as $f_{\mu\nu}f^{\mu\nu}$, which are probably generated upon coarse graining, should not affect the long-distance behavior of the theory since the Chern-Simons term involves only one derivative and renders the a_μ field massive.[15] The resulting partition function in this approximation is of the form

$$Z[A_\mu] = \int [d\phi\, da_\mu] e^{i(S_\phi[\phi,a_\mu,A_\mu] + S_a[a_\mu])} \tag{6}$$

with

$$\mathcal{L}_a = 4\theta\epsilon^{ij}(2a^0\partial_i a_j - a_i\partial_0 a_j) - (e^2/4\theta)\epsilon^{\mu\nu\sigma}a_\mu\partial_\nu a_\sigma, \tag{7}$$

$$\mathcal{L}_\phi = \phi^*[i\partial_0 - e(A_0 + a_0)]\phi - \tfrac{1}{2}\kappa\phi^*[-i\nabla - e(\mathbf{A}+\mathbf{a})]^2\phi + \mu|\phi|^2 - \lambda|\phi|^4, \tag{8}$$

and $\mu = 2\lambda n$, where n is the density.

This action is similar to the one introduced by Girvin.[5] There are some differences in that we have a $\lambda\phi^4$ term for the scalar field, and we have also incorporated time dependence in the action. The essential difference, however, is that Girvin's Chern-Simons action is for the sum of both the statistical gauge field and the real external electromagnetic field, while in our case, only the statistical gauge field appears in the Chern-Simons term. In Girvin's case, the partition function corresponding to (4) is independent of the external electromagnetic field, and therefore cannot be used to derive the fractional quantum Hall conductance. Furthermore, the vortex does not carry charge with respect to the real U(1) electromagnetic gauge group. In deriving (8) we have assumed that $V(r)$ is short ranged; if it is long ranged the expression can be generalized to include a renormalized interaction $\tilde{V}(r)$ which will be equal to $V(r)$ at long distances. In the spirit of the conventional Landau-Ginzburg theory, we ignore terms with higher powers of ϕ and higher derivatives that are generated by the coarse-graining procedure, and we treat κ and λ as phenomenological parameters. This completes the derivation of our model; we now proceed to demonstrate that this effective field theory correctly describes the phenomena related to the FQHE.

First consider the case $A_0 = 0$ and $\epsilon^{ij}\partial_i A_j = -B$ $=$const. It is immediately clear that S will be minimized by the trivial constant solution $\phi = \sqrt{n}$, $\mathbf{a} = -\mathbf{A}$, $a_0 = 0$. Since the statistical gauge field is related to the density via (3), this solution exists only for $\nu = n/n_B = \pi/\theta = 1/(2k-1)$. This does not necessarily mean that there is a solution only for the particular fraction corresponding to the θ chosen in the Lagrangian, since the different choices of k are connected via singular gauge transformations which induce an $\mathbf{r}$-dependent phase in ϕ.

To calculate the Hall conductance, we apply an external scalar potential A_0 with $\partial_i A_0 = -E_i$, in addition to the vector potential $\epsilon^{ij}\partial_i A_j = -B$. The observable (gauge invariant) current is given by

$$j^i = \frac{\delta S}{\delta A_i} = -\frac{\delta S_\phi}{\delta a_i} = +\frac{\delta S_a}{\delta a_i} = \frac{e^2}{2\theta}\epsilon^{ij}(\partial_0 a_j - \partial_j a_0), \tag{9}$$

where we used the equation of motion $\delta S/\delta a_i = 0$. Thus,

$$\langle j^i \rangle = \frac{1}{Z} \int [d\phi \, da_\mu] \frac{e^2}{2\theta} \epsilon^{ij} (\partial_0 a_j - \partial_j a_0) e^{iS(\phi, a_0, a_i, A_0, A_i)} . \tag{10}$$

By shifting the integration variable $a_0 = -A_0 + a_0'$, and expanding to linear order in the electric field, we find that the induced current is

$$\langle j^i \rangle_{\text{ind}} \equiv -i \left[\frac{\delta \ln Z}{\delta A_i} \bigg|_{A_0, A_i} - \frac{\delta \ln Z}{\delta A_i} \bigg|_{A_0 = 0, A_i} \right] = \frac{e^2}{2\theta} \epsilon^{ij} E^j = \sigma_H^{ij} E^j , \tag{11}$$

and since $\theta = \pi(2k-1)$, this demonstrates that the Hall conductance is quantized in odd fractions of e^2/h.

Let us now analyze what happens when we move away from the odd-integer filling fractions. Since the b field is locked at $eb = s2\pi n$, where now $n = n_B + \delta n$, the particles will feel a net field $e\,\delta b = e(b - B) = s2\pi\,\delta n$. Each particle (or hole) will acquire a cyclotron energy $= \kappa e\,\delta b/2$, which implies an energy density

$$\mathcal{E} = (2k-1)\pi e B \,|\delta n|\, \kappa . \tag{12}$$

For large B it is thus natural to assume that by moving away from the good filling fractions one creates localized density disturbances. As we shall see, our model exhibits such quasiparticle and quasihole excitations in the form of vortices similar to those found in Refs. 1, 2, and 5.

From the equation of motion derived from (6)–(8), one easily finds that at $n = \pi n_B/\theta$ there are static, nonuniform, finite-energy vortex solutions. If (r,φ) are polar coordinates with the center of the vortex at $r=0$, the $r \to \infty$ behavior of the solution is

$$\phi(r,\varphi) = \sqrt{n}\, e^{\pm i\varphi} , \tag{13}$$

$$\mathbf{a}(r,\varphi) = \pm \hat{\varphi}/er , \tag{14}$$

and $a_0(r,\varphi) = 0$, corresponding to one unit of statistical flux per vortex. The equation of motion for a^0 implies

$$j^0 \equiv -\frac{\delta S}{\delta A_0} = -\frac{\delta S_\phi}{\delta a_0} = \frac{\delta S_a}{\delta a_0} = \frac{e^2}{2\theta} b , \tag{15}$$

and so the total charge carried by the vortex is

$$q^1 = \int d^2x \, j^0 = \frac{e^2}{2\theta} \oint \mathbf{a} \cdot d\mathbf{r} = \pm \frac{\pi}{\theta} e = \pm \frac{e}{2k-1} . \tag{16}$$

These field configurations can thus be identified as the fractionally charged quasiparticle and quasiholes above the ground state. According to the results of Refs. 10–12, 17, and 18, this implies that the quasiparticles obey fractional statistics with $\theta_1 = q_1 \Phi_1 / 2 = \pi/(2k-1)$, where $\Phi_1 = 2\pi/e$ is the flux of the vortex. Note that since $\phi(\mathbf{r})$ must vanish at the center of the vortex, there is necessarily a difference in the profile and creation energies of the quasiparticles and quasiholes. (This also illustrates the intrinsic problems with the Landau-Ginzburg approach at distances of the order of the magnetic length, since the true quasiparticle density certainly does not vanish in the core.)

We have thus shown that the vortices in our model have the same charge and statistics as the quasiparticles in Laughlin's approach.[1,9,10] The presence of these excitations naturally leads to the so-called hierarchy scheme,[9,20] which has been proposed to explain the quantization of the Hall conductance at fractions other than $1/(2k-1)$.

Finally let us turn to the collective excitations. As a result of the symmetry-breaking potential in L_ϕ, the ϕ fields acquire a nonvanishing vacuum expectation value $|\langle\phi\rangle| = \sqrt{n}$. The ϕ field can thus be parametrized by $\phi(x) = [\phi_0 + \delta\phi(x)] e^{ie\eta(x)}$, $\mathbf{a}(x) = \delta\mathbf{a}(x) + \nabla\eta(x)$, and $a_0(x) = \delta a_0(x) - \partial_0 \eta(x)$, describing the amplitude and phase fluctuations about the classical vacuum. We see that the phase fluctuation $\eta(x)$ is "gauged away" in accordance with the standard Anderson-Higgs mechanism. Since the statistical gauge field is nondynamical, there is no propagating mode (massive or not) corresponding to phase fluctuations. This reflects that there is a unique ground-state in the quantum Hall effect and, we believe, implies that there will be no Josephson-type effects. Only the amplitude fluctuation remains, and by expanding the Lagrangian, up to terms quadratic in $\delta\phi$, $\delta\mathbf{a}$, and δa_0, about the constant solution, we find the following dispersion relation:

$$\omega(q)^2 = (e\kappa B)^2 + \tfrac{1}{4}\kappa q^2 (\kappa q^2 + 8\lambda\phi_0^2) . \tag{17}$$

Note that the mass of the amplitude mode is $\sim B$, and for negative λ the dispersion curve has the same shape as that derived in Refs. 7 and 8. Note that even for negative λ, as long as $|\lambda|/\kappa$ is sufficiently small, the quasiparticle creation energy is positive, and the Hamiltonian is bounded from below.

In conclusion, we have derived a field theory for the FQHE directly from the microscopic Hamiltonian, and find that a coarse-grained version of this theory describes almost all the known phenomenology of the FQHE including incompressibility, fractional Hall conductance with odd denominators, and the fractional charge and statistics of the quasiparticles. It is to be warned, however, that this coarse-grained theory certainly makes errors on the magnetic length scale, and it treats the statistical gauge field $\mathbf{a}$ within mean-field theory; i.e., the particles feel the b field which produces the statistics, whereas in fact the exact $\mathbf{a}$ is pure gauge. Despite these shortcomings, we believe that the long-wavelength properties of the quantum-Hall system are correctly reproduced by this Landau-Ginzberg theory. We note that the same physical idea, i.e., that the long-wavelength effects of the physical magnetic field are canceled by the statistical field, is the basic result of the cooperative-ring-exchange theory[2,3] of the quantum Hall effect, and of a mean-field theory recently introduced by Laughlin.[21] There are some similarities between the present results and some independent recent work of Read.[22]

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