Thermodynamics of the Classical Massive-Thirring-Sine-Gordon Model

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The proper classical limit $\hbar \to 0$ of the quantum Bethe-*Ansatz* thermodynamics gives a new set of integral equations for the thermodynamics of the classical massive-Thirring-sine-Gordon model. The theory contains the thermodynamic information about breathers and solitons, and reproduces the transfer-matrix-method result for arbitrary coupling constant.

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The thermodynamics of the sine-Gordon model described by the Hamiltonian

$$H=\frac{1}{\tau_0}\int dx\{\tfrac{1}{2}\phi_t^2+\tfrac{1}{2}\phi_x^2+m^2(1-\cos\phi)\} \tag{1}$$

has been developed along two independent lines: quantum theory and classical theory. The quantum thermodynamics has basically been settled by the Bethe-*Ansatz* (BA) method[1] and the factorized S-matrix method.[2] On the other hand, the classical theory has its own importance and beauty, and has been extensively studied by the soliton-gas phenomenology[3] and the transfer-matrix (TM) method.[4] The latter gives an exact free energy, whereas the former explores the soliton contribution to the thermodynamics which is beyond the scope of the TM method. The first connection between the quantum BA method and the classical TM method was made by the present author in the weak coupling limit where the quantum theory reduces to the classical theory: The leading term in the free energy calculated by the classical TM method was reproduced by the quantum BA method.[5] (See Sasaki,[6] Maki,[7] Chen, Johnson, and Fowler,[8] and Timonen *et al.*[9] for recent developments in the weak coupling regime.) On the other hand, Takayama and Ishikawa,[10] and Sasaki[11] recently developed a new formulation of the classical soliton-gas phenomenology. Although their study is important in that it is for an arbitrary coupling constant, their theory for calculating the free energy still contains a factor $2\pi\hbar$ originating from the quantization of particle momenta, and therefore it is not a genuine classical theory. The significance of the factor $2\pi\hbar$ and its intuitive treatment was first discussed by Theodorakopoulus[12] and Opper[13] in the context of the Toda lattice.

In this Letter, I shall study the proper classical limit $\hbar \to 0$ of the quantum BA thermodynamics, and make a complete connection between the quantum BA method and the classical TM method through a new set of coupled integral equations. The new equations are for the renormalized energies of solitons and breathers at finite temperatures, identifying the respective contributions of solitons and breathers to the free energy, and the calculated total free energy exactly agrees with that of the classical TM method for *arbitrary coupling constant*.

I work on the massive-Thirring model described by the Hamiltonian with an explicit $\hbar$ and $c=1$:

$$H=\int dx[-i\hbar(\psi_1^\dagger\partial_x\psi_1-\psi_2^\dagger\partial_x\psi_2)+m_0(\psi_1^\dagger\psi_2+\psi_2^\dagger\psi_1)+2g_0\psi_1^\dagger\psi_2^\dagger\psi_2\psi_1]\,. \tag{2}$$

With the identification[14]

$$g_0=16/\tau_0\,, \tag{3}$$

the sine-Gordon model (1) is equivalent to the fermion field theory (2). The complete analysis of the Hamiltonian (2), i.e., the Bethe wave function, physical vacuum, elementary excitations (solitons and breathers), and the renormalized two-body phase shifts between elementary excitations, is given in Ref. 1.

The Dirac $\hbar$ appears in the theory in the following manner. First, it appears in the energy spectrum of the breather

$$E_n(\alpha)=2M_s\sin[\tfrac{1}{2}n\pi(\pi/\mu-1)]\,,\quad n=1,2,\ldots,\frac{\pi}{\pi-\mu}-1\,, \tag{4}$$

through the constant μ,

$$\mu=-\cot^{-1}(g_0/2\hbar)\,. \tag{5}$$

M_s is the soliton mass and should be identified as the classical soliton mass $8/\tau_0$ in the limit $\hbar\to 0$. In the limit $\hbar \to 0$, $\mu\to\pi$ from below, and the mass spectrum of the breather becomes continuous with spacing $\sim\hbar$. It is noted that the Korepin excitations, exotic excitations in the quantum theory, disappear in the limit $\hbar\to 0$. This means that soliton and antisoliton behave exactly the same way, and therefore the contributions of antisolitons can be counted by simply doubling the contributions of solitons.

Second, it appears in the quantization of momentum

$$P_j(\alpha)=\frac{2\pi\hbar}{L}\times\text{integer}+\frac{1}{L}\sum_i\int d\alpha'\Delta_{ji}(\alpha'-\alpha)\rho_i(\alpha'), \tag{6}$$

where i and j run over breathers and solitons, Δ_{ji} denotes the two-body phase shift between i-kind and j-kind excitations, and ρ_i denotes the density distribution of the i-kind excitation in the rapidity space. In the limit $\hbar\to 0$, the rather intricate quantal phase shifts Δ take the following simple forms:

$$\frac{\partial}{\partial(\alpha/2)}\Delta_{ss}(\alpha)=\tfrac{1}{2}g_0\ln\left[\frac{\cosh(\alpha/2)+1}{\cosh(\alpha/2)-1}\right]\equiv\dot{\Delta}_{ss}(\alpha/2), \tag{7a}$$

$$\frac{\partial}{\partial(\alpha/2)}\Delta_{s\theta}(\alpha)=g_0\ln\left[\frac{\cosh(\alpha/2)+\sin\theta}{\cosh(\alpha/2)-\sin\theta}\right]\equiv\dot{\Delta}_{s\theta}(\alpha/2), \tag{7b}$$

$$\frac{\partial}{\partial(\alpha/2)}\Delta_{\theta\theta'}(\alpha)=g_0\ln\left[\frac{\cosh(\alpha/2)-\cos(\theta+\theta')}{\cosh(\alpha/2)+\cos(\theta+\theta')}\frac{\cosh(\alpha/2)+\cos(\theta-\theta')}{\cosh(\alpha/2)-\cos(\theta-\theta')}\right]\equiv\dot{\Delta}_{\theta\theta'}(\alpha/2), \tag{7c}$$

where for the nth breather in the spectrum (4), I have defined a new variable θ as

$$\theta=\lim_{\hbar\to 0}\tfrac{1}{2}n\pi(\pi/\mu-1). \tag{8}$$

Here we can see the first connection to the classical theory. From (4), (8), and the fact that $\mu\to\pi$ from below, we find that

$$0\le\theta\le\pi/2, \tag{9}$$

and (4) reduces to the classical energy spectrum of the breather. As for the classical two-body phase shifts, I note that a quantal entity which continuously develops to the classical solitary wave in the limit $\hbar\to 0$ is not the quantized solitary wave itself, but its wave packet with a group velocity

$$\frac{\partial\omega}{\partial k}=\frac{\partial\omega/\partial\alpha}{\partial k/\partial\alpha}. \tag{10}$$

With this in mind, I have directly calculated the two-body phase shifts from the Hirota multisoliton solution.[15] With appropriate analytic continuations in the rapidity space, the calculated phase shifts agree with the derivatives (7), not Δ themselves as is expected, except for a field-theoretic renormalization $\alpha\to\alpha/2$, which can be removed by the rescaling $\alpha/2\to\alpha$ in the final integral equations (15) and (16) below.

Finally in the limit $\hbar\to 0$, the classical volume element Ω in phase space is related to the number of corresponding quantum states Φ by

$$\Omega=(2\pi\hbar)^N\Phi, \tag{11}$$

where N denotes the number of associated degrees of freedom. Working in the particle number representation with particle number densities ρ and associated hole densities $\tilde{\rho}$ in the rapidity space, the number N is identified as

$$N=2L\int d\alpha\,\rho_s(\alpha)+2L\sum_{i=\text{breathers}}\int d\alpha\,\rho_i(\alpha), \tag{12}$$

where the factor 2 in the first expression accounts for the contribution of antisolitons, whereas that in the second expression is due to the fact that a breather has an internal degree of freedom as well as a translational degree of freedom. In this way, the classical free energy $F=E-TS$ contains an additional term which does not exist in the quantal expression [cf. (4.2) and (4.4) of Ref. 1, Boltzmann's constant = 1)]:

$$-NT\ln(2\pi\hbar). \tag{13}$$

Correspondingly, due to the continuous availability of phase space in the classical thermodynamics, the ratio $\tilde{\rho}/\rho$ behaves like $\sim 2\pi\hbar$ for solitons and $\sim(2\pi\hbar)^2$ for breathers. Therefore the ϵ functions of Yang and Yang[16] in the present case should be introduced as follows:

$$\tilde{\rho}_s(\alpha)/\rho_s(\alpha)=2\pi\hbar\exp[\epsilon_s(\alpha)/T], \tag{14a}$$

$$\tilde{\rho}(\theta,\alpha)/\rho(\theta,\alpha)=(2\pi\hbar)^2\exp[\epsilon(\theta,\alpha)/T]. \tag{14b}$$

The above arguments justify and extend an intuitive procedure of Theodorakopoulos[12] and Opper[13] for the Toda lattice.

Minimizing the free energy with respect to variations of densities ρ_i [note that (6) relates $\tilde{\rho}$ to ρ], and taking the classical limit $\hbar\to 0$ with all the above arguments, I reach the following set of integral equations:

$$\epsilon_s(\alpha)=M_s\cosh\alpha+2T\int d\alpha'\dot{\Delta}_{ss}(\alpha'-\alpha)\exp[-\epsilon_s(\alpha')/T]+2\pi g_0T\int_0^{\pi/2}d\theta\int d\alpha'\dot{\Delta}_{s\theta}(\alpha'-\alpha)\exp[-\epsilon(\theta,\alpha')/T], \tag{15a}$$

$$\epsilon(\theta,\alpha)=2M_s\sin\theta\cosh\alpha+2T\int d\alpha'\dot{\Delta}_{s\theta}(\alpha'-\alpha)\exp[-\epsilon_s(\alpha')/T]+2\pi g_0T\int_0^{\pi/2}d\theta\int d\alpha'\dot{\Delta}_{\theta\theta'}(\alpha'-\alpha)\exp[-\epsilon(\theta',\alpha')/T]. \tag{15b}$$

In terms of the solutions of (15), the free energy is given by

$$\frac{F}{L}=-2T\int d\alpha\, M_s \cosh\alpha \exp[-\epsilon_s(\alpha)/T]-2\pi g_0 T\int_0^{\pi/2} d\theta \int d\alpha\, 2M_s \sin\theta\cosh\alpha\exp[-\epsilon(\theta,\alpha)/T]\,. \tag{16}$$

The free energy (16) should be compared with the TM free energy[4,6,17]

$$\frac{\tilde{F}}{L}=-\frac{T}{l}\ln\left[2\pi l T\left(\frac{e}{\pi}\right)\right]+\frac{1+a_0/2q}{\tau_0}\,, \tag{17}$$

where $q\equiv(2/\tau_0 T)^2$ and a_0 is the smallest eigenvalue of the Mathieu equation

$$\varphi''+(a_0-2q\cos 2x)\varphi=0\,. \tag{18}$$

The lattice constant l is related to the cutoff $\infty \rightarrow \alpha_m$ in the integral equations (15) and (16) by $\sinh\alpha_m=\pi/l$.

Here an important remark is due on (16) and (17). That is, they are not mathematically equivalent because (16) is negative definite, whereas (17) can be positive when the quantity $\zeta\equiv T/\cosh\alpha_m \ll 1$. We must note, however, that both formulas (16) and (17) are expected to provide an accurate classical free energy only in the temperature regime

$$\zeta \gtrsim 1\,. \tag{19}$$

In the case of the Toda lattice, where only a solitonlike excitation exists, Opper[13] proved a rigorous mathematical equivalence between a BA free energy and the Toda exact free energy.[18] In the massive-Thirring-sine-Gordon model, in contrast to the Toda lattice, we have breathers as well as solitons, and I find that the sum over the breather spectrum and the classical limit $\hbar\rightarrow 0$ do not commute in the nonclassical temperature regime, giving rise to the above mentioned lack of rigorous mathematical equivalence between (16) and (17).

This can be seen most clearly in the weak coupling limit $\tau_0\rightarrow 0$ or $g_0\rightarrow\infty$. In this limit, the soliton contribution disappears, and (15b) and (16) reduce to, after the rescaling $g_0\theta\rightarrow\theta$,[19]

$$\epsilon(\theta,\alpha)=\theta\cosh\alpha+8\pi^2 T\int_0^\infty d\theta' \min(\theta,\theta')\exp[-\epsilon(\theta',\alpha)/T]\,, \tag{15b$'$}$$

$$F/L=-2\pi T\int d\alpha \cosh\alpha\int_0^\infty d\theta\,\theta\exp[-\epsilon(\theta,\alpha)/T]\,. \tag{16$'$}$$

This is the procedure which first takes the limit $\hbar\rightarrow 0$ and then the sum over the breather spectrum. If we sum over the breather spectrum first, we get[5,20]

$$F/L=(T/2\pi)\int d\alpha\cosh\alpha\ln[1-\exp(-\hbar\cosh\alpha/T)]-(K/L)T\ln(2\pi\hbar)\,, \tag{20}$$

where the second term comes from (11) with K denoting the number of the lattice points. In the limit $\hbar\rightarrow 0$, (20) becomes mathematically identical with the TM expression (17) in the weak coupling limit $\tau_0\rightarrow 0$.

My last task, therefore, is to show the equivalence between Eqs. (15) and (16) and Eqs. (17) and (18) in the classical temperature regime $\zeta\gtrsim 1$ for *arbitrary coupling constant* τ_0. I have examined two typical cases: (i) the weak coupling limit $\tau_0\rightarrow 0$, $T=8$, and $\alpha_m=\ln(6T)/2$, and (ii) $\tau_0=1$, $T=8$, and $\alpha_m=\ln(6T)$. Case (i) is examined by (15b′) and (16′), whereas case (ii) by (15) and (16). In both cases, the calculated free energies agree with those of (17) and (18) within a numerical error ($\lesssim 1\%$ relative error).

To summarize, a new set of integral equations is derived for the thermodynamics of the classical massive-Thirring-sine-Gordon model for an arbitrary coupling constant, and is shown to reproduce the transfer-matrix-method result in the classical temperature regime. Details of the present paper along with further analysis of the basic equations will be reported elsewhere.

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