Elastic Scattering of Solitary Waves in the Strongly Dissipative Toda Lattice

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The behavior of solitary waves has been studied both experimentally and numerically in a strongly dissipative Toda lattice. In the absence of dissipation the system is an integrable soliton system, but due to dissipation the solitary wave decreases, and develops a tail. Numerical simulation of the head-on collision of two solitons with different amplitudes indicates that the scattering is nevertheless purely elastic. This suggests that certain parts of soliton theory should be applicable in systems where the wave itself is not conserved.

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Soliton theory has found many applications in real systems, which shows that solitons are robust structures, and that deviations from ideal situations (e.g., weak dissipation) can be handled using perturbation theory. In this Letter we report results on the behavior of solitons in a strongly dissipative Toda lattice. In this system the solitary wave deforms markedly from the ideal soliton as it travels in the lattice. This decay might suggest that the system is so far from the ideal Toda lattice that the scattering of solitons will also be highly nonelastic. Numerical simulations presented below show, however, that the scattering is elastic (with a typical small phase shift) within numerical accuracy.

In the Toda lattice the nearest-neighbor interaction is exponential. It has been shown to be integrable,[1,2] and in particular it has soliton solutions and an infinite number of conserved quantities. Several studies have also been made on the solitons in a lattice with weak dissipation.[3-5] In this paper we analyze the strongly dissipative Toda lattice.

The Toda lattice can be realized experimentally by a nonlinear electrical transmission line, as proposed first by Hirota and Suzuki.[6] Adding both a serial resistance R and parallel conductance G we get the dissipative LC lattice shown in Fig. 1.[4] The network equations describing this transmission line are

$$\begin{aligned} L\,\partial_t I_n + R I_n &= V_{n-1} - V_n\,, \\ \partial_t Q_n + G V_n &= I_n - I_{n+1}\,, \end{aligned} \qquad (1)$$

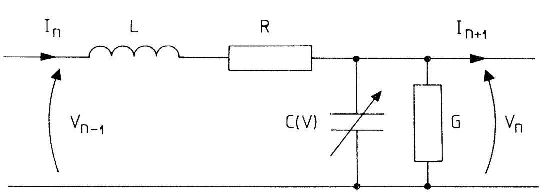

FIG. 1. A section of the nonlinear transmission line with resistance R and conductance G.

and the nonlinear charge is given by

$$Q_n = Q_0 \ln(1 + V_n/F_0)\,. \qquad (2)$$

Q_0 and F_0 are parameters specifying the nonlinear capacitor.

Equation (1) can be transformed to

$$\partial_\tau^2 \ln(1+v_n) + \tilde{R}\,\partial_\tau \ln(1+v_n) + \tilde{G}\,\partial_\tau v_n + \tilde{R}\tilde{G} v_n = v_{n-1} + v_{n+1} - 2v_n\,, \qquad (3)$$

where $v_n = V_n/F_0$, $\tau = (F_0/LQ_0)^{1/2} t$, $\tilde{R} = (R/L)(LQ_0/F_0)^{1/2}$, and $\tilde{G} = (G/C_0)(LQ_0/F_0)^{1/2}$. If $\tilde{R} = \tilde{G} = 0$, Eq. (3) is equivalent to the Toda lattice, and has soliton solutions

$$v_n = \sinh^2(\omega)\,\mathrm{sech}^2[\sinh(\omega)t \pm \omega n]\,. \qquad (4)$$

We have studied the system (3) by solving it numerically using the four-step Adams-Bashfort-Moulton method. Even the dissipative system has conserved

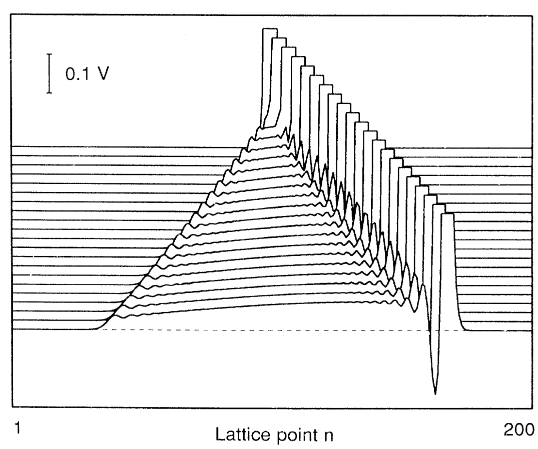

FIG. 2. The time evolution of an initial state given by an ideal one-soliton solution with $R = 20\ \Omega$, $G = 0$. The initial state is at the top. For clarity we have cut away the tops of the curves.

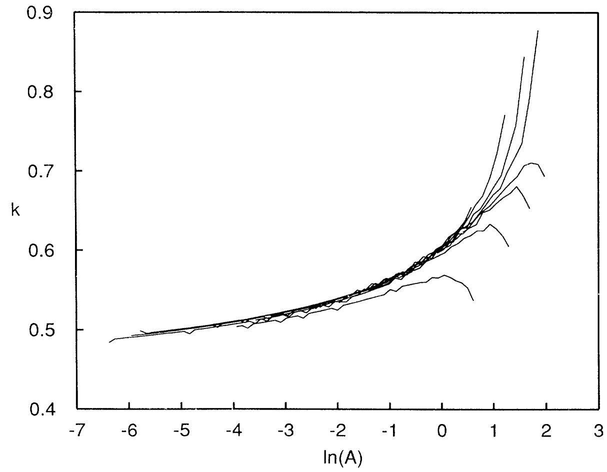

FIG. 3. The logarithmic derivative of the soliton amplitude k as a function of the logarithm of the amplitude A for different initial amplitudes and widths ($R=20\ \Omega$, $G=0$). The initial states for the curves starting from the top were somewhat narrower than the ideal solitons (4).

quantities; they are $J_1=\sum_{n=-\infty}^{\infty}Q_n$, when $G=0$, or $J_2=\sum_{n=-\infty}^{\infty}I_n$, when $R=0$. Computational errors were checked by comparing theoretical and numerical values of these quantities, and the errors were always smaller than 0.3%. The values of the parameters were $L=22$ μH, $Q_0=583$ pC, and $F_0=6.2$ V (these values were also used in our previous experimental studies[7,8]). The number of lattice sites was 200.

Let us first look at the behavior of a single soliton. Figure 2 shows how the soliton changes as a result of the dissipation. The solution is shown at 21 different moments of time, with a time interval of $3.2(F_0/LQ_0)^{1/2}$.

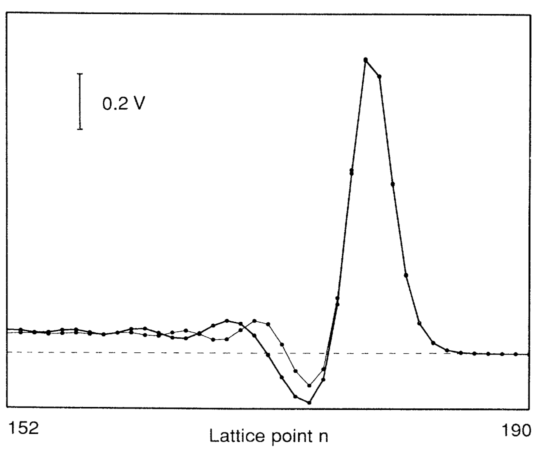

FIG. 4. Two one-soliton solutions with different initial amplitudes $A=4.0$ V (thin line) and $A=8.0$ V (thick line) recorded at a moment when they have the same amplitude. The taller soliton had traveled 3348 ns and the smaller 2374 ns.

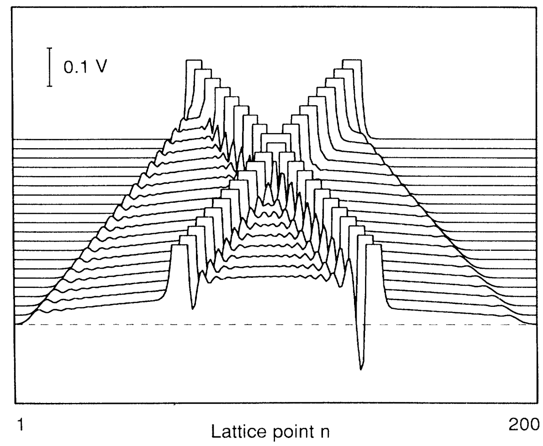

FIG. 5. The time evolution of an initial state given by an ideal two-soliton solution with $R=20\ \Omega$, $G=0$.

The initial state was an ideal one-soliton solution (4) centered around the lattice point $n=100$ and with the initial amplitude $A\ [=F_0\sinh^2(\omega)$ in (4)$]=6.0$ V. (The dissipation was caused by $R=20\ \Omega$, $G=0$.) As the soliton travels a positive tail appears behind it. There is also a definite dip after the soliton. The same properties have been observed in experiments.[8] We have found a good approximate solution for the tail using simple linear methods.[9]

The amplitude of the main soliton part decreased to 19% of the original after it traveled sixty lattice points. To characterize the soliton decay further we have calculated the logarithmic derivative of the soliton amplitude $k=(L/R)d_tA/A$.[8] Results are shown in Fig. 3.

An important observation is that the curves starting with different initial amplitudes (but with the same resistance) seem to approach a common curve rather quickly.

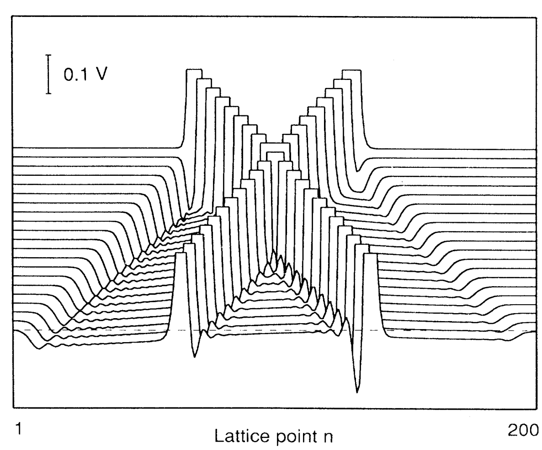

FIG. 6. The time evolution of an initial state given by an ideal two-soliton solution with $R=0$, $G=8.54\times10^5\ \Omega^{-1}$.

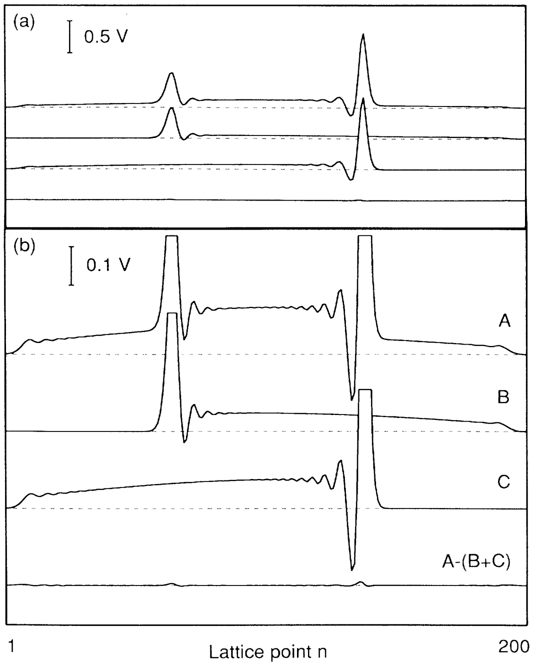

FIG. 7. (a) A two-soliton solution after the collision (A), two one-soliton solutions (B and C), and the result after the subtraction $A-(B+C)$, $R=20\ \Omega$, $G=0$. (b) Same as in (a) but the vertical scale magnified by a factor of 6.

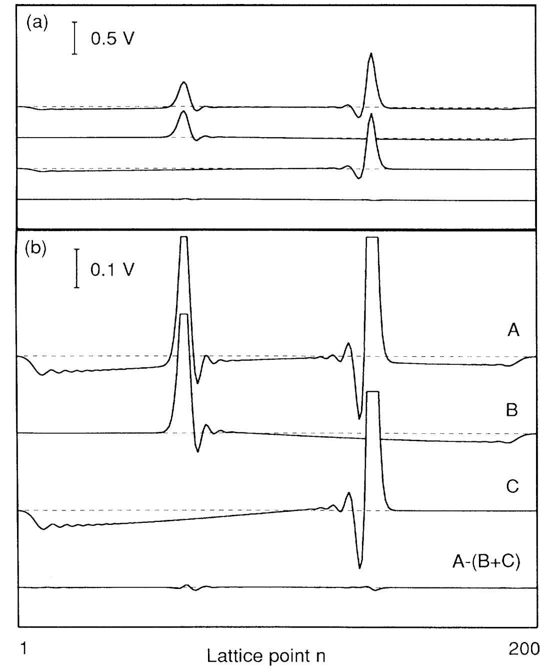

FIG. 8. Same as Fig. 7 but $R=0$, $G=8.54\times10^{-5}\ \Omega^{-1}$.

This suggests that there are some definite decaying soliton solutions (which does not depend on the fine details of the initial conditions) which attract all initial shapes. This can be seen clearly in Fig. 4, where two solutions with different initial amplitudes are plotted together. The elapsed time was chosen so that the highest two v_n values of the solitons would be the same. The soliton parts are identical although the tails are different.

For integrable systems it is well known that multisoliton solutions are attractors to all initial configurations. It is interesting that this phenomenon survives the relatively strong dissipation. Apparently the time scale for attraction is still smaller than the time scale for decay.

Let us next consider what happens in a collision. In our numerical simulation of head-on collisions the initial state was a sum of two ideal one-soliton solution (this is a good starting point because the solitons have practically no overlap), with the greater soliton ($A=6.0$ V) centered at the lattice point $n=70$ and the smaller ($A=2.0$ V) at $n=130$. Figure 5 illustrates how the initial state develops as a function of time in the case of $R=20\ \Omega$, $G=0$. The tails overlap after the collision. In the case of $R=0$, $G=8.54\times10^{5}\ \Omega^{-1}$ the tails are negative as shown in Fig. 6. The strength of dissipation is in both cases so strong that the pulse amplitudes decreased to 14%–26% of the original value after the pulses traveled sixty lattice points.

Figures 7 and 8 show that the collision is purely elastic. The top curves are the final configurations from Figs. 5 and 6, the two curves in the middle describe how a single pulse would have evolved in the same time

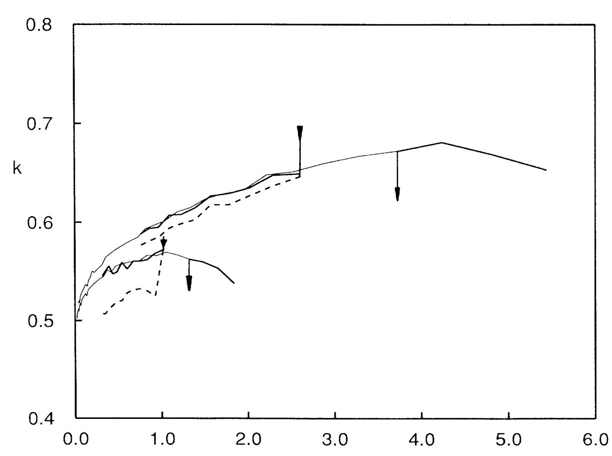

FIG. 9. The logarithmic derivative of the soliton amplitude k as a function of the amplitude A ($R=20\ \Omega$, $G=0$) for the colliding solitons (thick line) with tail corrections (the behavior after collision without tail corrections is given by the dashed line). The decay of amplitude follows closely the behavior of the single soliton (thin line). The values of k during collision are not shown.

period, and the bottom curve shows the result after subtracting the two one-soliton solutions from the two-soliton solution. The computed single pulses were initially located slightly off the starting points of the colliding solitons in order to account for the phase shift. The last curve is practically zero everywhere and shows that the solitons scatter elastically within numerical accuracy.

We have calculated the logarithmic derivative of the soliton amplitude also in the case of the collision as shown in Fig. 9. Before and after the collision the amplitude dependence of the k quantity is the same as for the one-soliton solution, if the amplitude of the tail is subtracted from the amplitude of the soliton (cf. Fig. 3).

The numerical results presented here show that certain properties of exactly solvable soliton systems persist even when the system is strongly dissipative. Of course we have to give up the idea of a permanent traveling wave, but many interesting properties remain: We have shown that there is some (decreasing) traveling wave which attracts all initial conditions, and that the collision of two waves is elastic. The extension of analytical methods to cover this situation will be an important problem.

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