Phase Coherence and Nonequilibrium Behavior in Josephson Junction Arrays

K. K. Mon
Department of Physics and Astronomy and Center for Simulational Physics, University of Georgia, Athens, Georgia 30602

S. Teitel
Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627

(Received 7 October 1988)

We model a two-dimensional periodic array of Josephson junctions in a transverse magnetic field by the uniformly frustrated XY model. We report the results of extensive nonequilibrium simulations of the array within the resistively-shunted-junction model. I-V curves are computed for the unfrustrated and fully frustrated cases, and consistency with recent experiments is found. For the fully frustrated case, new theoretical ideas are presented which show the Ising degrees of freedom to dominate nonequilibrium behavior near T_c.

PACS numbers: 74.60.Ge, 64.60.−i, 74.50.+r, 85.25.Dq

A two-dimensional periodic array of Josephson junctions in a uniform transverse magnetic field provides a direct experimental realization[1] of the class of uniformly frustrated XY models.[2,3] Such models display a rich range of critical behavior, including a Kosterlitz-Thouless (KT) transition[4] in the unfrustrated case, and a combined KT-Ising transition in the fully frustrated case.[3(a)] Theoretical work[3,5] has focused primarily on the equilibrium properties of these models. Experimental studies[6,7] make nonequilibrium measurements, particularly current-voltage (I-V) curves, and resistivity. In this paper we present a theoretical analysis of the nonequilibrium case, based on numerical simulations and simple stability arguments. The interaction of excitations of the ordered, field-induced, vortex structures present in equilibrium, with an applied external current, results in resistance due to net vortex motion in the nonequilibrium driven state. Understanding this phenomenon is of great fundamental interest, as well as important for making closer contact with experiment.

We consider a two-dimensional square $N \times N$ array of lattice constant a, with a Josephson junction on each bond. Along one edge of the array ($x=0$) a uniform current I is injected into each node, while along the opposite edge ($x=N$) a current I is extracted from each node. Periodic boundary conditions are taken in the y direction. The pseudo-Hamiltonian is

$$\mathcal{H} = -J_0 \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j - A_{ij}) - J \sum_{i_y} (\theta_{(i_x=0,i_y)} - \theta_{(i_x=N,i_y)}) . \quad (1)$$

The first term in (1) gives the usual uniformly frustrated XY model[3]: θ_i is the phase of the superconducting node at site $i=(i_x,i_y)$, $\langle ij \rangle$ is the junction formed by nearest-neighbor nodes i,j, $A_{ij} = (2e/\hbar c)\int_i^j \mathbf{A} \cdot d\mathbf{l}$ is the integral of the vector potential from node i to j, and the bare critical current of an isolated junction is $I_0 = (2e/\hbar) J_0$. The A_{ij} obey the constraint that the sum around any unit cell of the array is constant

$$A_{ij} + A_{jk} + A_{kl} + A_{li} = 2\pi f ,$$

where $f = Ha^2/\Phi_0$ is the number of flux quanta Φ_0 of external magnetic field H per unit cell of the array. The unfrustrated case is $f=0$. The fully frustrated case is $f=\frac{1}{2}$. The second term in (1) gives the effects of the external current, $I=(2e/\hbar)J$, making (1) the analog for arrays of the "washboard potential," commonly used for a single junction.[8] Minimization of $\mathcal{H}$ with respect to the θ_i gives supercurrent conservation at each node, the requirement for a time-independent solution to the dynamics. Defining the phases to have values $-\infty < \theta_i < +\infty$, this term makes $\mathcal{H}$ unbounded, with ever lower minima at increasing θ_i. Thermal activation of the system over barriers separating equivalent local minima of $\mathcal{H}$, differing only by shifts of 2π in the θ_i, will provide the mechanism for an average rate of phase slippage $d\Delta\theta/dt$, and hence voltage V, across the array.

We assume the resistively shunted-junction (RSJ) model for the dynamics of an individual junction[8]

$$V_{ij} = \frac{\hbar}{2e} \frac{d(\theta_i - \theta_j)}{dt} = R_n [I_{ij} - I_0 \sin(\theta_i - \theta_j - A_{ij}) + \eta_{ij}] , \quad (2)$$

where V_{ij} is the voltage across junction $\langle ij \rangle$, R_n is the normal resistance, I_{ij} is the total normal plus super current through the junction, and η_{ij} is a thermal noise current with

$$\langle \eta_{ij}(t) \eta_{ij}(t') \rangle = (2T/R_n) \delta(t-t') ,$$

with T the temperature. I_0 and R_n are implicit functions of temperature and magnetic field, and these dependencies must be known for detailed comparison with experimental junctions. The RSJ model should be valid when capacitance effects may be ignored, as in the proximity coupled junctions of Ref. 7(b). It may be more questionable for tunnel junctions as in Ref. 7(a).

Following Shenoy,[9] we sum (2) over all junctions exiting a given node. Inverting the resulting matrix equation gives the equations of motion for the θ_i

$$\frac{\hbar}{2e}\frac{d\theta_i}{dt} = R_n \sum_j G'_{ij}\left[\frac{2e}{\hbar}\frac{\partial \mathcal{H}}{\partial \theta_j} + \sum_{\hat{\mu}} \eta_{j,j+\hat{\mu}}\right], \qquad (3)$$

where $\mathcal{H}$ is as in (1) and G'_{ij} is the two-dimensional lattice Green's function with periodic boundary conditions in the y direction and free boundary conditions in the x direction. As $\sum_i \theta_i$ is a constant of motion, the singular piece of the Green's function, corresponding to a uniform rotation of all phases, has been subtracted out in the construction of G'.[9] For large separations, $G'_{ij} \simeq (1/2\pi)\ln(|\mathbf{r}_i - \mathbf{r}_j|/a)$. The variables $\hat{\mu}$ range over all directions exiting a given node.

We have numerically simulated[10] the stochastic equations of motion (3) as a function of temperature T, and applied current I, for both the unfrustrated ($f=0$) and the fully frustrated ($f=\frac{1}{2}$) cases. For each run we have computed the average voltage drop V across a unit cell of the array

$$V/R_n = I - I_0 \langle\langle \sin(\theta_i - \theta_{i+\hat{x}} - A_{i,i+\hat{x}})\rangle\rangle, \qquad (4)$$

where $\langle\langle \cdots \rangle\rangle$ is a combined thermal average and average over all junctions on the horizontal bonds of the lattice (the direction of the external current). The equation of motion was integrated with discrete time steps of $\Delta t \simeq 0.05(2eR_nI_0/\hbar)$. $N_t \simeq 10^5$ time steps were used to compute averages. Both Δt and N_t were varied to verify that steady state was achieved. Arrays of length $N=6$–16 were studied, and for the range of current reported, finite-size effects appear negligible for the largest arrays. Standard block averaging and independent runs were used to estimate sampling errors. We have been able to study voltages over 3 orders of magnitude, $V/R_nI_0 \sim 1 \rightarrow 0.005$, with typical errors of order 0.0025. Calculations were done on a vectorized Cyber 205, with computation time growing as N^4. For $N=16$ a typical run took 0.7 CPU hours.

The resulting I-V curves for $f=0$ and $\frac{1}{2}$ are shown in Figs. 1(a) and 1(b). For $f=0$, theoretical arguments[11] predict a power-law relation for $T < T_c$,

$$V \sim I^{1+a(T)}, \quad a(T) = \pi \Upsilon(T)/T, \qquad (5)$$

where $\Upsilon(T)$ is the helicity modulus, or renormalized spin-wave stiffness of the XY model. In Fig. 2 we plot $a(T)$ for $f=0$. Solid circles are results from power-law fits to our data in Fig. 1(a). We estimate an error of 25% in these fits, due to statistical error in computed V. The dashed line is the theoretical prediction (5) using numerical results for $\Upsilon(T)$ from the equilibrium simulations of Ref. 3(a). A jump in $a(T)$ from the value 2 to 0 at T_c results from the universal discontinuous jump in $\Upsilon(T_c)$, characteristic of the KT transition of $f=0$.[11] In any experiment or simulation, this jump will be smeared out due to finite current effects. Agreement with theory has been observed experimentally,[7] and as seen in Fig. 2, our simulation results are consistent.

Following recent experimental analysis,[7] we have also fitted our I-V data for $f=\frac{1}{2}$ to a power-law form. Our results are shown as squares in Fig. 2, and are seen to lie within the scatter of the experimental data from van Wees, van der Zant, and Mooij[7(a)] (triangles) and Carini[7(b)] (open circles). $a(T)$ appears to go continuously to zero at T_c, and disagrees with Eq. (5) (shown as

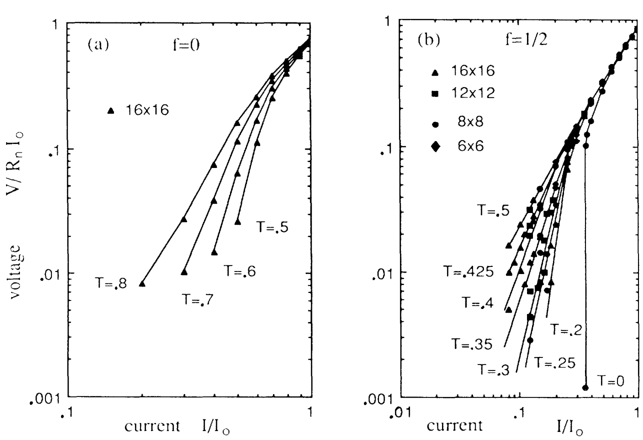

FIG. 1. Numerically computed I-V curves for the Josephson junction array for the (a) $f=0$ and (b) $f=\frac{1}{2}$ cases, as functions of temperature k_BT/J_0. Solid lines are guides to the eye only.

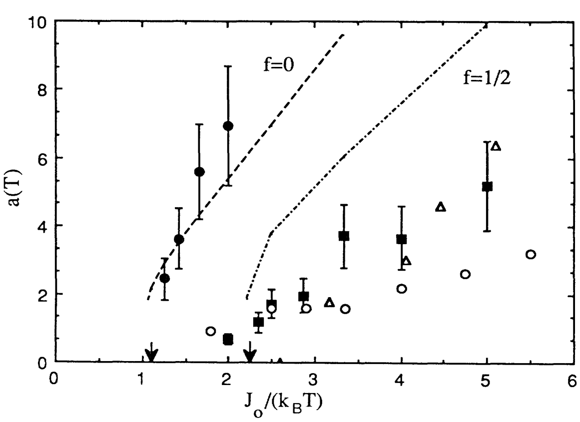

FIG. 2. Exponent $a(T)$ of the current-voltage relation $V \sim I^{1+a}$ vs J_0/T. For the unfrustrated case: solid circles are results from fits to the data of Fig. 1(a); dashed line is the theoretical prediction (5). For the fully frustrated case: squares are results from fits to the data of Fig. 1(b); dot-dashed line is from Eq. (5); triangles are experimental results of van Wees, van der Zant, and Mooij [Ref. 7(a)]; open circles are experimental results of Carini [Ref. 7(b)]. Arrows indicate transition temperatures $T_c(f=0)$ and $T_c(f=\frac{1}{2})$. Estimated errors of 25% are shown as error bars on our numerical results.

dot-dashed line), where now Υ is taken to be the helicity modulus of the fully frustrated model.[3(a)]

The lack of agreement with (5) for the fully frustrated model can be understood in terms of the following theoretical arguments. The uniformly frustrated XY model with external current corresponds to an equivalent Coulomb gas problem, in a uniform external electric field oriented perpendicular to the applied current. The Coulomb gas Hamiltonian is

$$\mathcal{H}_{\mathrm{Cg}} = -2\pi^2 J_0 \sum_{ij} n_i G'_{ij} n_j - 2\pi J \sum_i n_i \mathbf{r}_i \cdot \hat{\mathbf{y}} , \tag{6}$$

where $n_i \equiv m_i - f$ are fractional charges and $m_i = 0, \pm 1, \pm 2, \ldots$. Neutrality $\sum_i n_i = 0$ is imposed and G' is again the lattice Green's function. The first term in (6) is the usual Coulomb gas.[2] The second term is a dipole interaction between the charges n_i and a uniform electric field.[11] The helicity modulus Υ of the original XY model is related to the dielectric function ϵ of the Coulomb gas[12] by $\Upsilon(T)/J_0 = \epsilon^{-1}(T)$.

For the fully frustrated case, $f = \frac{1}{2}$, at *zero current* $J = 0$, the doubly degenerate ground state is a lattice of alternating $n_i = +\frac{1}{2}, -\frac{1}{2}$ charges. At low temperatures the model possesses quasi-long-range order in the phase θ_i, with finite Υ, and long-range order in a chirality variable that measures the ordering of the ground-state charge lattice $\chi = (1/N^2)\sum_i n_i (-1)^{x_i + y_i}$. At a temperature T_c the helicity modulus goes to zero with a discontinuous jump, $\Upsilon(T_c)/k_B T_c \gtrsim 2\pi$. The chirality $\langle\chi\rangle$ vanishes with Ising-type critical behavior[3(a)] at a temperature $T_I \gtrsim T_c$. Numerical[13] and theoretical[14] arguments indicate T_I is extremely close or equal to T_c.

At finite current $I = (2e/\hbar)J$, the low-temperature ordered phase becomes unstable with respect to two different types of excitations. The first, Fig. 3(a), we call KT type. It consists of the interchange of a given pair of $+\frac{1}{2}, -\frac{1}{2}$ charges with separation $\mathbf{r}$. For large $\mathbf{r}$, the free energy for such an excitation is

$$F_{\mathrm{pair}}(\mathbf{r}) = \frac{2\pi}{\epsilon(T)} (J_0 \ln|\mathbf{r}/a| - J\mathbf{r}\cdot\hat{\mathbf{y}}) . \tag{7}$$

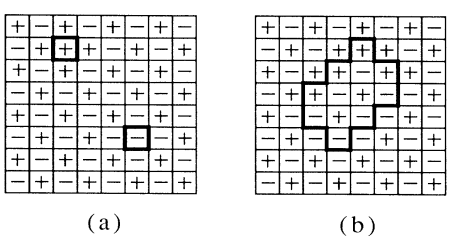

FIG. 3. Instabilities of the fully frustrated XY model with finite current. (a) KT-type excitations are formed by the interchange of a single pair of charges, and are unstable for separations $r > r_c = J_0/J$. (b) Ising-type excitations are domains formed by flipping nearest-neighbor pairs in the direction perpendicular to the applied current, and are unstable for sizes $L > L_c \sim \epsilon\sigma/\pi J\langle\chi\rangle$.

A pair oriented in the y direction with separation $r > r_c = J_0/J$ will decrease its free energy by increasing r and unbinding. The rate for such pair unbinding is

$$W_{\mathrm{pair}} \sim e^{-F_{\mathrm{pair}}(r_c)/k_B T} \sim J^{2\pi\Upsilon/k_B T} . \tag{8}$$

In Ref. 11(a) it was shown that the unbinding and recombination of pairs gives rise to a density of free charges $n_f \sim (W_{\mathrm{pair}})^{1/2}$. Making the analogy between the charges in the Coulomb representation and vortices in the phase representation, we have that the resistivity due to pair unbinding is $R = V/I \sim n_f \sim J^{\pi\Upsilon/k_B T}$, the origin of Eq. (5).

A second instability results from excitations of Fig. 3(b), which we call Ising type. These consist of a domain of the oppositely ordered charge lattice, within the ordered phase. When the domain is formed by interchanging all $+\frac{1}{2}, -\frac{1}{2}$ nearest-neighbor pairs in the same direction parallel to $\hat{\mathbf{y}}$, a domain of area L^2 will have a net dipole moment $\langle\chi\rangle L^2$ with respect to the ordered phase. The free energy to create such a domain is therefore

$$F_{\mathrm{Ising}}(L) \sim \sigma(T) 4L - 2\pi J\langle\chi\rangle L^2/\epsilon , \tag{9}$$

where $\sigma(T)$ is the surface tension between the two ordered phases. The system is unstable for domains with $L > L_c = \sigma\epsilon/\pi J\langle\chi\rangle$. Once a domain reaches a size $L > L_c$ it is energetically favorable for it to grow, filling the entire array, and resulting in a net shift of the ground-state charge lattice by one site, with a reversal in the sign of $\langle\chi\rangle$. The Josephson array will have a net shift of vortices perpendicular to the applied current, resulting in a net voltage. The rate for nucleating such critical domains is

$$W_{\mathrm{Ising}} \sim e^{-F_{\mathrm{Ising}}(L_c)/k_B T} \sim e^{-4\sigma^2\epsilon/2\pi J\langle\chi\rangle k_B T} . \tag{10}$$

One would thus expect a contribution to the voltage from these Ising excitations $V \sim W_{\mathrm{Ising}}$. It is problematic, however, to fit our numerical results to the form (10). For too small current J, finite lattice effects and finite simulation time prevent an accurate computation of voltage (when $V/R_n I_0 \lesssim 0.005$). For too large $J \sim J_c$, where $J_c \simeq 0.35 J_0$ is the $T = 0$ critical current of the array,[15] σ has a strong anisotropic dependence on J as the metastable minima of $\mathcal{H}$ (1) vanish. For large J the isolated domain nucleation picture may also become invalid. The exact range of J where (10) should hold is not clear, even for the simpler analogous case of the nearest-neighbor Ising model in a magnetic field.[16] However, measurements of the chirality $\langle\chi\rangle$ during our simulations do indicate the Ising excitations to be playing a considerable role for the range of J of our numerical data. For all applied currents which produced a statistically clear nonzero voltage, $\langle\chi\rangle$ was observed to flip sign over the course of the simulation with a rate increasing with J. Thus the nucleation of these Ising domains causes the system to switch between the two ground states of the system, destroying the Ising order present in equilibrium, after a sufficiently long time.

Although the above considerations make difficult an expression for the I-V relationship in the fully frustrated model, the following general features must remain true. For a fixed temperature T, as current I is decreased the pair mechanism must eventually dominate as the algebraic rate (8) will decay more slowly than the exponential rate (10). However, for fixed I, as $T \rightarrow T_c^-$, a temperature will eventually be reached at which the Ising mechanism msut dominate, since $\Upsilon(T_c^-)$ and hence $F_{\text{pair}}(r_c)$ remains finite, while

$$F_{\text{Ising}}(L_c) \sim \sigma^2(T)/\langle\chi\rangle \sim |T-T_c|^{2\nu-\beta} \rightarrow 0$$

for the two-dimensional Ising model. Thus for any experimental measurement at finite I, it will be the Ising excitations that dominate the nonlinear I-V curves sufficiently close to T_c. The prediction (5) will no longer hold and thus it will be difficult to extract information about the discontinuous jump in $\Upsilon(T_c^-)$ from such measurements.

Although our simulations have been based on the particular dynamics of the RSJ model, the energetic stability arguments above should have more general validity. Thus even for arrays in which capacitance may not be ignorable, one would expect to see significant differences between the unfrustrated and fully frustrated cases, due to the crucial role of the Ising-type excitations.

K.K.M. wishes to thank the Advanced Computational Method Center of the University of Georgia for partial support. S.T. wishes to thank the Dean of the College of Arts and Science of the University of Rochester for partial support.

[1]C. J. Lobb, Physica (Amsterdam) **126B**, 319 (1984).

[2]J. Villain, J. Phys. C **10**, 1717, 4793 (1977); E. Fradkin, B. A. Huberman, and S. H. Shenker, Phys. Rev. B **18**, 4789 (1978).

[3](a) S. Teitel and C. Jayaprakash, Phys. Rev. B **27**, 598 (1983); (b) S. Teitel and C. Jayaprakash, Phys. Rev. Lett. **51**, 1999 (1983).

[4]J. M. Kosterlitz and D. Thouless, J. Phys. C **6**, 1181 (1973); J. V. Jose, L. P. Kadanoff, S. Kirkpatrick, and D. R. Nelson, Phys. Rev. B **16**, 1217 (1977).

[5]W. Y. Shih and D. Stroud, Phys. Rev. B **30**, 6774 (1984); T. C. Halsey, J. Phys. C **18**, 2437 (1985); M. Y. Choi and S. Doniach, Phys. Rev. B **31**, 4516 (1985); S. E. Korshunov, J. Stat. Phys. **43**, 17 (1986).

[6]R. F. Voss and R. A. Webb, Phys. Rev. B **25**, 3446 (1982); R. A. Webb, R. F. Voss, G. Grinstein, and P. M. Horn, Phys. Rev. Lett. **51**, 690 (1983); D. W. Abraham, C. J. Lobb, M. Tinkham, and T. M. Klapwijk, Phys. Rev. B **26**, 5268 (1982); D. Kimhi, F. Leyvraz, and D. Ariosa, Phys. Rev. B **29**, 1487 (1984); R. K. Brown and J. C. Garland, Phys. Rev. B **33**, 7827 (1986).

[7](a) B. J. van Wees, H. S. J. van der Zant, and J. E. Mooij, Phys. Rev. B **35**, 7291 (1987); (b) J. P. Carini, Phys. Rev. B **38**, 63 (1988).

[8]M. Tinkham, *Introduction to Superconductivity* (McGraw-Hill, New York, 1975).

[9]S. R. Shenoy, J. Phys. C **18**, 5163 (1985).

[10]Similar simulations have been reported for the $f=0$ case: J. S. Chung and D. Stroud, Bull. Am. Phys. Soc. **33**, 341 (1988).

[11](a) V. Ambegaokar, B. I. Halperin, D. R. Nelson, and E. D. Siggia, Phys. Rev. B **21**, 1806 (1980); (b) B. I. Halperin and D. R. Nelson, J. Low Temp. Phys. **36**, 599 (1979).

[12]T. Ohta and D. Jasnow, Phys. Rev. B **20**, 139 (1979); P. Minnhagen and G. G. Warren, Phys. Rev. B **24**, 2526 (1981).

[13]S. Miyashita and H. Shiba, J. Phys. Soc. Jpn. **53**, 1145 (1984); D. H. Lee, J. D. Joannopoulos, J. W. Negele, and D. P. Landau, Phys. Rev. B **33**, 450 (1986); B. Berge, H. T. Diep, A. Ghazali, and P. Lallemand, Phys. Rev. B **34**, 3177 (1986); J. M. Thijssen and H. J. F. Knops, Phys. Rev. B **37**, 7738 (1988).

[14]M. Yosefin and E. Domany, Phys. Rev. B **32**, 1778 (1985); E. Granato and J. M. Kosterlitz, Phys. Rev. B **33**, 4767 (1986).

[15]The value of J_c depends on the manner in which the current is injected and removed at the ends. For uniform injection, as assumed here, J_c is somewhat lower than was obtained with a different boundary condition [Ref. 3(b)].

[16]K. Binder, Rep. Prog. Phys. **50**, 783 (1987).