Persistent Currents in Mesoscopic Rings and Cylinders

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(Received 15 August 1988)

Persistent currents in multichannel normal-metal rings threaded by a magnetic flux are treated by a Green's-function approach. The diffusive region is analyzed in detail and the typical current predicted as a function of disorder, temperature, and number of channels. The amplitude and temperature sensitivity of the effect are found to be governed by correlations in the energy spectrum of the ring, of range E_c proportional to the Thouless energy. The results are relevant to persistent-current experiments in mesoscopic metal rings.

PACS numbers: 72.10.Bg, 71.30.+h, 73.50.Bk, 73.60.Aq

The intriguing question of persistent currents in *normal-metal* rings enclosing a magnetic flux was first discussed in the 1960's.[1] The work by Büttiker, Imry, and Landauer[2] in 1983, predicting persistent currents in one-dimensional disordered loops, renewed the interest in the topic.[3-9] This interest is heightened by recent advances in submicrometer physics[10] that have brought the effect into reach of experimental investigation. The crucial question is, how disruptive are disorder and temperature to persistent currents in real normal-metal rings? Is the effect big enough to be of interest? The question is addressed here from the viewpoint of mesoscopic phenomena[10,11] with use of a new Green's-function approach.

We review some basic properties.[1-5] The electron eigenenergies E_n of a ring that encloses an area of magnetic flux ϕ are periodic functions of ϕ with period $\phi_0=hc/e$. An electron in a state E_n carries a current I_n that can be calculated from $I_n=-c\,\partial E_n/\partial\phi$. The total current is the sum over the contributions I_n of all states weighted with the appropriate occupation probability $I=\sum_n f(E_n)I_n$. The long-time average of the total current is a property of the thermodynamic state of the system and hence does not decay. It is periodic in the flux with period ϕ_0. The effect requires phase coherence of the electron wave function around the ring. A sizable current exists only for rings of small size. For example, the maximal current amplitudes for free electrons in perfect one-dimensional loops and thin-walled (short) cylinders of circumference L are $I_0=ev_F/L$ and $M^{1/2}I_0$, respectively, where M denotes the number of transverse channels. The characteristic temperature separating low- and high-temperature behavior is, in both cases, $k_BT^*=\Delta_1/2\pi^2$, where $\Delta_1=2\pi\hbar v_F/L$ is the one-channel level spacing at the Fermi surface.[6] At first sight, this result is surprising because the level spacing in M-channel rings is much smaller, namely, $\Delta_M \propto \Delta_1/M$. The explanation is that strong level correlations exist in the energy spectrum, which for perfect M-channel rings are of range $M\Delta_M$.[6]

The details of the persistent-current behavior depend on the relative magnitudes of various lengths and energies. We consider only *short* three-dimensional cylinders, for which the circumference L is much larger than the length L_y and wall width L_z, $L\gg L_y,L_z$. The number of transverse channels is $M=Ak_F^2/4\pi$, where $A=L_yL_z$. We focus on the regime of *diffusive* electron transport defined by $l_{\text{el}}<L<\xi$, in terms of the elastic mean free path l_{el} and localization length ξ. For an M-channel cylinder, $\xi\propto Ml_{\text{el}}$. Ballistic and strongly localized behavior occur for $L<l_{\text{el}}$ and $L>\xi$, respectively. We restrict the discussion to the *mesoscopic* region $L\lesssim l_\varphi$, where l_φ is the phase coherence length of the electron. For convenience, we assume that the electrons in the cylinder move in a field-free space and that the self-inductance of the ring is zero. Our calculations are for free-electron and tight-binding models.

Our principal finding is that, for the diffusive regime of metal rings, both the amplitude of the persistent current at $T=0$ and its temperature sensitivity are determined by correlations in the energy spectrum, of range $E_c=\pi^2\hbar D/L^2$. We find it instructive to express E_c as $E_c\propto M_{\text{eff}}\Delta_M\propto\Delta_1 l_{\text{el}}/L$, in terms of the level spacing, Δ_M, and the number of active channels as function of disorder,[12] $M_{\text{eff}}=Ml_{\text{el}}/L$. (Roughly, M_{eff} changes from M to order 1 over the diffusive regime, $l_{\text{el}}<L<\xi$.) Note that E_c is also proportional to the Thouless energy V_{Th},[13] and determines the range of correlations in the conductance correlation function.[14] This result places the persistent-current problem into close context with the problems of Aharonov-Bohm and conductance fluctuations in wires and rings studied previously.[10,11] For the observability of persistent currents in normal-metal rings, it is significant that characteristic energy is E_c, rather than the much smaller Δ_M.

Specifically we find the following: (i) The amplitude of the persistent current is determined by the root-mean-square (rms) current $\langle I^2\rangle^{1/2}$, which we call typical current. Averages are over disorder configurations. $\langle I^2\rangle^{1/2}$ is proportional to $l_{\rm el}/L$ as a function of disorder, and is independent of the number of channels. In contrast, the average current amplitude falls off exponentially with disorder, as $\exp(-L/2l_{\rm el})$, but increases as $M^{1/2}$ with channel number. The sign of the average current is practically random for different microscopic configurations of disorder. Such large fluctuations are characteristic of mesoscopic phenomena.[7] (ii) At finite temperatures, the value of the thermal diffusion length $l_T=(2\pi\hbar D/k_BT)^{1/2}$ determines the crossover from low- $(L<l_T)$ to high- $(L>l_T)$ temperature behavior of the typical persistent current. The corresponding characteristic energy is $E_c=\pi^2\hbar D/L^2$. When $k_BT>E_c$, the amplitude of the typical current is reduced by $\exp[-\mathrm{const}(k_BT/E_c)^{1/2}]$. (iii) The typical single-level current decreases with increasing number of channels M. Correlations in the energy spectrum over E_c are such that the ratio of the typical total and single-level currents is proportional to $M_{\rm eff}^{1/2}$. (iv) For a given microscopic configuration of disorder, the total persistent current $I_{\rm total}$ as function of chemical potential μ changes sign with average period E_c, with fluctuations in the period of order of its size. This also implies that the range of correlations in the current-current correlation function is E_c. The traces of $I_{\rm total}$ vs μ are "sample specific." (v) The typical total and single-level currents measure the sensitivity of the system to changes in the boundary conditions. Therefore, our results allow inferences about the Thouless formula for the conductivity.[13] In the following we derive these results.

We express the persistent current in terms of Green's functions. This facilitates the process of our taking ensemble averages over disorder. Our new formula for the persistent current is

$$I=\int_{-\infty}^{\infty}\frac{dE}{2\pi i}\sum_k f(E)[G^+(k,k,E)-G^-(k,k,E)]I_k^{(0)}\,. \qquad (1)$$

Here k labels the discrete eigenstates of the perfect system (in the absence of disorder), $I_k^{(0)}$ are the associated single-level currents, $G^{\pm}$ denote the advanced and retarded one-particle Green's function, and $f(E)$ is the Fermi-Dirac distribution. Equation (1) is easily derived. Expand the eigenstates of the disordered ring in terms of those of the perfect ring $\psi_k^{(0)}$, $\psi_n=\sum_k a_{nk}\psi_k^{(0)}$. Since the current operator is diagonal in the unperturbed basis, the single-level currents I_n can be expressed as a sum over the $I_k^{(0)}$, $I_n=\sum_k |a_{nk}|^2 I_k^{(0)}$, where $|a_{nk}|^2$ is the probability of overlap between the states ψ_n and $\psi_k^{(0)}$. The total current is the sum over the contributions I_n of all states ψ_n weighted with the occupation probability $f(E)$. The one-particle Green's function of a finite-sized ring is $G(k,k';E)=\sum_n a_{nk}a_{nk'}^*/(E-E_n)$, with a discrete spectrum. Expressing the overlap probabilities $|a_{nk}|^2$ in terms of the residues of G, one obtains (1). Formula (1) holds for one microscopic configuration of disorder. The calculation of the (disorder) average current $\langle I\rangle$ and the $\mathrm{rms}(I)=\langle I^2\rangle^{1/2}$, therefore, requires as input the disorder averaged one- and two-particle Green's functions, $\langle G(k,k;E)\rangle$ and $\langle G(k,k;E)G(k',k';E')\rangle$, respectively.

Consider the structure of Eq. (1). The system geometry and flux enter the calculation through $I_k^{(0)}$ and k. The latter index labels the discrete eigenstates of the perfect system, e.g., for free electrons in a thin-walled cylinder $k_x=2\pi(n+\phi/\phi_0)/L$, $k_y=\pi m/L_y$, and $k_z\approx\pi m'/L_z$, with $n=0,\pm1,\pm2,\ldots$ and $m,m'=1,2,\ldots$.[1,6] The effects due to disorder and corresponding length and energy scales enter via the pole structure of $\langle G\rangle$ and $\langle GG\rangle$. It is possible to replace the latter averages by their standard bulk forms (except for the discrete label k). We approximate $\langle G^{\pm}\rangle\approx 1/(E-E_k\mp iB)$.[15] Then $\langle|a_{nk}|^2\rangle$ has Lorentzian form with width B and height $\Delta_M/\pi B$. We find that to obtain the dominant contributions to the typical current, $\langle GG\rangle$ can be approximated by $\langle G\rangle^2\langle K\rangle\langle G\rangle^2$. The leading contributions are due to the diffusion and Cooperon poles in $\langle K^{+-}\rangle$ and $\langle K^{-+}\rangle$, such as

$$\tau_{\rm el}^{-1}|V_{k,k'}|^2/[-(i/\hbar)(E-E')+D(k\mp k')^2]\,,$$

for small energies and momenta.[15] Diagrammatically, both the ladder and maximally crossed diagrams are included. In terms of $\tau_{\rm el}$ or $l_{\rm el}=v_F\tau_{\rm el}$, the broadening B and diffusion constant D are given by $B=\hbar/2\tau_{\rm el}$ and $D=v_F^2\tau_{\rm el}/d$, respectively. Note that in dimensionless units $2\pi B/\Delta_1=L/2l_{\rm el}$ and $2\pi^2k_BT/\Delta_1=T/T^*$ with $k_BT^*=\Delta_1/2\pi^2$. To leading order, $L/2l_{\rm el}$ and T/T^* determine the amplitude of the average current, cf. Eq. (4). The poles in $\langle K^{+-}\rangle$ introduce the characteristic energy $E_c=\pi^2\hbar D/L^2$. In dimensionless units $E_c/\Delta_1\propto l_{\rm el}/L$ and $[k_BT/(2\pi\hbar D/L^2)]^{1/2}=L/l_T$, where $l_T=(2\pi\hbar D/k_BT)^{1/2}$. To leading order, $l_{\rm el}/L$ and L/l_T determine the amplitude of the typical current, cf. Eqs. (2) and (3).

Our results and conclusions are as follows. For a single-ring experiment, the typical current is given by $\langle I^2\rangle^{1/2}$. When $l_{\rm el}\ll L<\xi$, it exhibits power-law behavior as function of disorder, $l_{\rm el}/L$. At $T=0$,

$$\langle I^2\rangle^{1/2}=\pm\left(\frac{8}{3d\pi^2}\right)^{1/2}\left(\frac{l_{\rm el}}{L}\right)I_0\sin\left(\frac{2\pi\phi}{\phi_0}\right)+\text{higher harmonics}\,. \qquad (2)$$

There is no dependence on the number of channels and no significant (e.g., oscillatory) dependence on chemical potential. The current amplitude can also be written as the product of the conductance $g/(2e^2/h)=4M_{\rm eff}/3$ times the inverse density of states at the Fermi surface, Δ_M. We note, in the energy integral leading to (2) appears an exponential factor that effectively cuts off the integral of E_c, or $M_{\rm eff}$ levels, below the Fermi surface.

This looks as if the last $M_{\rm eff}$ levels determine the sign and magnitude of the total current. At temperatures $k_B T > E_c$ (or $L > l_T$), but $k_B T \ll B$, there are two regions of slightly different temperature behavior. In the first ($L > l_T$, L_y^2/l_T, and L_z^2/l_T),

$$\langle I^2 \rangle^{1/2} = \pm \left[\frac{16\pi}{3d} \right]^{1/2} \left[\frac{l_{\rm el}}{L} \right] \left[\frac{L}{l_T} \right]^{3/2} \exp\left[-\frac{\pi L}{l_T} \right] I_0 \sin\left[\frac{2\pi\phi}{\phi_0} \right] + \text{higher harmonics}. \tag{3}$$

While for higher temperatures, when $L \ll L_y^2/l_T$ and/or $L \ll L_z^2/l_T$, the prefactor should be multiplied by $[2L_y/(Ll_T)^{1/2}]^{1/2}$ and/or $[2L_z/(Ll_T)^{1/2}]^{1/2}$, respectively.

The amplitude of the average current $\langle I \rangle$ decreases exponentially with disorder and it increases with the number of channels M. For $l_{\rm el} < L$, at $T = 0$,

$$\langle I \rangle = \frac{2}{\pi} \left[\frac{M}{L/l_{\rm el}} \right]^{1/2} I_0 \exp\left[-\frac{L}{2l_{\rm el}} \right] \sin\left[\frac{2\pi\phi}{\phi_0} \right] + \text{higher harmonics}. \tag{4}$$

At $T > T^*$, this expression is multiplied by $2T/T^*$ and the $L/l_{\rm el}$ are replaced by $L/l_{\rm el} + 2T/T^*$. Note that the energy scale is Δ_1 for both the disorder and temperature dependences. The higher harmonics are reduced by correspondingly higher powers of the exponential factors. The total average current exhibits oscillatory dependence on the chemical potential with period Δ_1. The result (4) is the rms amplitude of this fluctuation. The current amplitude in (4) is proportional to the probability amplitude that the electron is not scattered while traversing the ring once. For a two-dimensional cylinder, the exponent of the $l_{\rm el}/L$ prefactor is $\frac{1}{4}$.

The amplitude of the typical single-level current $\langle I_n^2 \rangle^{1/2}$ decreases with increasing number of channels M. By choosing a contour in the energy integrals that encloses on average only one state, we find, e.g., at $\phi/\phi_0 = 0.25$,

$$\langle I_n^2 \rangle^{1/2} = \pm \left[\frac{l_{\rm el}}{6ML} \right]^{1/2} I_0 . \tag{5}$$

In contrast to $\langle I^2 \rangle^{1/2}$, the single-level current depends only weakly on ϕ with a typical amplitude given by (5). The typical single-level current can also be expressed in terms of the total average transmission amplitude between any two channels, $\langle I_n^2 \rangle^{1/2} \propto \langle | t_{ij}(E_n) |^2 \rangle^{1/2} I_0$.

The ratio between the typical total and single-level currents is proportional to $M_{\rm eff}^{1/2}$, as implied by (2) and (5). Numerical data for small two-dimensional cylinders are consistent with this result. It may suggest the interpretation that the typical total current is the rms superposition of $M_{\rm eff}$ typical single-level currents.

From the results for the currents we infer information about the energy spectrum and correlations. Consider the total current $I_{\rm total}$ for a short cylinder with a *fixed* microscopic configuration of disorder. Our results for the properties of the typical current suggest strongly that $I_{\rm total}$ exhibits sign changes with an average period $E_c \propto M_{\rm eff}\Delta_M \propto \Delta_1 l_{\rm el}/L$, with fluctuations in the period of the order of its size. In other words, each sample exhibits a sample-specific, aperiodic trace of $I_{\rm total}$ vs μ. This in turn implies strong correlations among the slopes of the energy levels $E_n(\phi)$, to make possible this oscillation. One indication in support of the conjecture comes from the high-temperature properties of the typical current, Eq. (3). The result $\exp[-\text{const}(k_B T/E_c)^{1/2}]$ implies that groups of single-level currents with alternating sign are separated at a scale of E_c and that this separation fluctuates as a function of μ and from sample to sample. Note also that a change in μ by E_c causes a phase change of π in a typical wave function along a diffusive path around the ring, $\delta k_F v_F \tau_D = \pi$. (For a perfect ring, $I_{\rm total}$ oscillates with period Δ_1.) We see the aperiodic oscillations of the total current numerically in small samples.

We comment on the Thouless formula for the conductance. The typical currents, being related to the derivative of the energy with respect to flux, are a measure of the sensitivity of the eigenstates of the system to changes in the boundary conditions. Using $\Delta E = \phi_0 \partial E_n/\partial\phi$ as the definition of the boundary sensitivity of the eigenstates, we deduce from Eq. (5) that $G \propto (\Delta E/\Delta_M)^2$, previously derived[16] in one dimension. Defining alternatively the flux sensitivity in terms of that of the total energy, $\Delta E = \phi_0 \partial E/\partial\phi$, we conclude from Eq. (2) that then the conductance formula has power one, $G \propto \Delta E/\Delta_M$.[13]

There are many crossover phenomena that can be studied within this framework. (i) Geometrical crossover to thin-walled long cylinders ($L \ll L_y$). We find an enhancement of the amplitudes in (2), (3), and (4) by factors proportional to $(L_y/L)^{1/2}$, as a result of additional correlations. The same was found for currents in perfect long cylinders.[6] (ii) Crossover to the strongly localized regime. In the strongly localized regime, $\xi < L$, we find $\langle I^2 \rangle^{1/2} \propto I_0 M^{-1} \exp(-L/2\xi)$, as expected because there the overlap of the wave functions around the ring is exponentially small. (iii) Crossover to very high-temperature behavior for $\langle I^2 \rangle^{1/2}$. The temperature corrections in Eq. (3) are for moderate temperatures. At $k_B T \gg B$ (or $l_T \ll l_{\rm el}$) another crossover occurs to temperature behavior proportional to $\exp(-T/T^*)$. We find this result also in simulations.

The formulas offer guidance for the optimal choice of parameters to observe persistent currents. Here, we dealt with small metal rings, for which M is large. (The case of small M can be realized by use of small semiconductor rings.) The size of the persistent current depends crucially on a favorable ratio of L_{el}/L. Equations (2) and (3) hold deep in the diffusive regime. At weaker disorder in the diffusive regime, also M dependence comes to bear and the typical current is larger than predicted by (2) and (3). Then one may use (4) to estimate a lower bound to the typical current. The temperature requirements, namely to satisfy $l_T > L$ (or $k_B T < E_c$) and to assure large phase coherence lengths l_φ, seem to be compatible. For a metal ring ($k_F \approx 1.2\times 10^{10}$ m^{-1}, e.g., Au or Cu) of $L = 1$ μm and $L_y L_z = 0.02\times 0.02$ μm^2, one finds $M \approx 0.5\times 10^4$, $I_0 \approx 0.2$ μA, and $T^* \approx 3$ K. The following estimates are for flux $\phi/\phi_0 = 0.25$ and contain a factor 2 for spin. Assuming a large $l_{el} = 0.1$ μm (0.2 μm), one obtains for the typical current from Eq. (4) a lower bound 0.04 μA (0.7 μA), while Eq. (2) yields the estimates 0.01 μA (0.03 μA), which are too low as expected. Deep in the diffusive region, if we assume $l_{el} = 0.02$ μm, Eq. (2) yields an estimate of 3 nA. For the latter case, $\pi L/l_T \lesssim 1$ requires temperatures $T \lesssim 50$ mK. For experiments that use an ensemble of many macroscopically identical rings, each ring carries a current of average magnitude $\langle I^2 \rangle^{1/2}$ and random sign. Period halving is not expected.[5]

Several effects neglected in the above analysis may make it more difficult to observe the persistent current. The magnetic field penetrating the ring will smear out the perfect ϕ_0 periodicity, depending on the aspect ratio of the ring and the strength of the field. Finite phase coherence l_φ (e.g., because of inelastic scattering and magnetic impurities) reduces the amplitude of the effect. Self-inductance effects are small due to the small size of the currents. The question of thermal noise in the ring needs further study.

In conclusion, we have presented a detailed analysis of the persistent current in multichannel normal-metal rings in the diffusive region of electronic transport. We expect the leading dependences on the dimensionless parameters l_{el}/L, L/l_T, $M^{1/2}$, $L/2l_{el}$, and T/T^* to be model independent, but not the coefficients. Long cylinders, not discussed here in detail, offer a further enhancement of the current.

We are particularly grateful to C. Castellani, C. DiCastro, D. J. Thouless, and R. A. Webb for helpful and stimulating discussions. We have benefited from conversations with M. Büttiker, D. P. DiVincenzo, O. Entin-Wohlman, Y. Imry, R. Landauer, P. A. Lee, A. D. Stone, and others. H.F.C. and E.K.R. acknowledge with thanks the warm hospitality of C. Castellani and C. Di Castro at the Universty of Rome. Y.G. acknowledges support by the U.S.-Israel Binational Science Foundation and the Minerva Foundation, Munich, Germany. This work was supported in part by the National Science Foundation under Grant No. DMR-85-09392.

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[1]For example, N. Byers and C. N. Yang, Phys. Rev. Lett. **7**, 46 (1961); F. Bloch, Phys. Rev. **137**, A787 (1965); F. Bloch, Phys. Rev. **166**, 415 (1968); R. Landauer, unpublished; M. Schick, Phys. Rev. **166**, 401 (1968); L. Gunther and Y. Imry, Solid State Commun. **7**, 1391 (1969); Y. Imry and L. Gunther, unpublished; and F. Bloch, Phys. Rev. B **2**, 109 (1970).

[2]M. Büttiker, Y. Imry, and R. Landauer, Phys. Lett. **96A**, 365 (1983).

[3]R. Landauer and M. Büttiker, Phys. Rev. Lett. **54**, 2049 (1985).

[4]M. Büttiker, Phys. Rev. B **32**, 1846 (1985), and in *New Techniques and Ideas in Quantum Measurement Theory,* edited by D. M. Greenberger [Ann. N.Y. Acad. Sci. **480**, 194 (1986)].

[5]H. F. Cheung, Y. Gefen, E. K. Riedel, and W. H. Shih, Phys. Rev. B **37**, 6050 (1988).

[6]H. F. Cheung, Y. Gefen, and E. K. Riedel, IBM J. Res. Develop. **32**, 359 (1988).

[7]For preliminary reports of the results (i), see E. K. Riedel, H. F. Cheung, and Y. Gefen, Bull. Am. Phys. Soc. **33**, 367 (1988), and Phys. Scr. (to be published).

[8]N. Trivedi and D. A. Browne, Phys. Rev. B **38**, 9581 (1988).

[9]O. Entin-Wohlman and Y. Gefen, unpublished.

[10]S. Washburn and R. A. Webb, Adv. Phys. **35**, 375 (1986); also D. E. Prober, Microelectron. Eng. **5**, 203 (1986).

[11]For reviews emphasizing theory, see Y. Imry, in *Directions in Condensed Matter Physics,* edited by G. Grinstein and G. Mazenko (World Scientific, Singapore, 1986), p. 101; A. G. Aronov and Y. V. Sharvin, Rev. Mod. Phys. **59**, 755 (1987); and A. D. Stone, in *Proceedings of the Second International Symposium on Foundations of Quantum Mechanics, Tokyo, 1986,* edited by M. Namiki *et al.* (Physical Society of Japan, Tokyo, 1987), p. 207.

[12]Y. Imry, Europhys. Lett. **1**, 249 (1986).

[13]J. T. Edwards and D. J. Thouless, J. Phys. C **5**, 807 (1972); also D. J. Thouless, Phys. Rev. Lett. **39**, 1167 (1977).

[14]P. A. Lee and A. D. Stone, Phys. Rev. Lett. **55**, 1622 (1985); A. D. Stone and Y. Imry, Phys. Rev. Lett. **56**, 189 (1986).

[15]See, e.g., D. Vollhardt and P. Wölfle, Phys. Rev. B **22**, 4666 (1980).

[16]P. W. Anderson and P. A. Lee, Prog. Theor. Phys. Suppl. (Japan) **69**, 212 (1980).