Existence of Anderson Localization of Classical Waves in a Random Two-Component Medium

C. M. Soukoulis,[1,2] E. N. Economou,[1,3] G. S. Grest,[1] and M. H. Cohen[1]
[1]*Corporate Research Science Laboratory, Exxon Research and Engineering Company, Annandale, New Jersey 08801*
[2]*Ames Laboratory and Department of Physics, Iowa State University, Ames, Iowa 50011*
[3]*Research Center of Crete and Department of Physics, University of Crete, Iraklion, Crete, Greece*
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An exact mapping of the classical wave problem to that of electronic motion is utilized together with extensive numerical results to examine the question of the existence of genuine localization (i.e., one occurring when both components have real positive dielectric constants) of classical waves in random binary alloys $A_{1-x}B_x$. We find that scalar waves do exhibit localization. We have also developed a coherent potential approximation which for $x < 0.2$ gives results not that much different from the numerical ones. This result can be easily generalized to electromagnetic fields as well.

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Until recently, most of the work on localization has been focused on electronic systems; however, within the last three years the question of classical wave localization (CWL)[1-11] has received attention, partly because the CWL offers both the potential for a direct check of localization theory as well as the possibility of other localization characteristics distinct from those of electrons.[1] Several successful experiments on the enhanced coherent backscattering (a precursor of Anderson localization) of light have been reported.[3-6] Strong localization has not been observed yet.[7,8] The theoretical results[9-11] on the existence of CWL are still not definite, since they are very sensitive to the various calculational approximations. It is the purpose of this Letter to make a quantitative analysis of the phenomena of CWL by a random array of scatters. First, we make an exact mapping of the scalar localization problem to the electronic one, which has been studied extensively within the last ten years. The resulting picture gives clearly the range of parameters most promising to search for Anderson localization in the CWL problem.

For the CWL problem, we consider a composite material consisting of two components with dielectric constants ϵ_1 and ϵ_2. We will only consider cases where ϵ_1 and ϵ_2 are real, i.e., cases where there is no absorption. We also assume $0 < \epsilon_1 < \epsilon_2$, but we discuss briefly what happens when ϵ_1 becomes negative. The concentration of the ϵ_1 component is $1-x$. The wave equation satisfied by the scalar-wave amplitude u at frequency ω is given by

$$\nabla^2 u + \frac{\omega^2}{c^2}\epsilon u = 0 , \tag{1}$$

where c is the wave velocity for $\epsilon = 1$ and ϵ is the dielectric constant which is a random variable.

In the corresponding electronic problem, the wave equation is given by

$$\nabla^2 u \frac{2m}{\hbar^2}(E-V)u = 0 , \tag{2}$$

where E is the energy under consideration and V is the random potential experienced by the electron. In analogy with the CWL problem we have that V takes the values E_A and E_B $(E_A > E_B)$ with probabilities $1-x$ and x, respectively, and $\delta \equiv E_A - E_B$.

In Fig. 1, the electronic behavior is summarized in the energy-disorder plane. The δ axis, $E - E_A = 0$, represents the bottom of the spectrum of the pure A material; for $E - E_A > 0$ both materials are locally of propagating nature. The straight line OB, given by $\delta = -(E - E_A)$ or, equivalently, by $E = E_B$, represents the bottom of the pure-B spectrum and the lower bound of the spectrum of the composite random material, the so-called Lifshitz

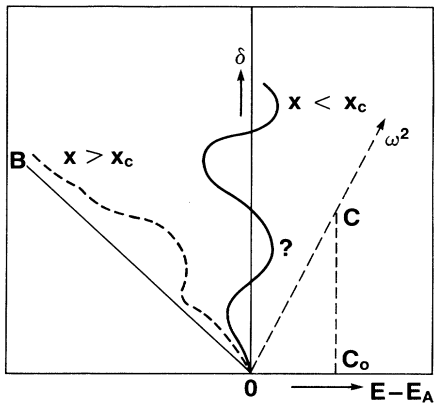

FIG. 1. Schematic diagram for a composite binary random electronic system $A_{1-x}B_x$; E is the energy and $\delta = E_A - E_B \geq 0$, where E_A (E_B) is the bottom of the spectrum of pure A (B) material. The line OB corresponds to $E = E_B$. The expected mobility edge trajectory, separating localized states to its left from extended states to its right, is also shown either for $x < x_c$ (heavy solid line) or for $x > x_c$ (heavy dashed line); x_c is the critical percolation concentration for the B material. The slope of the line OC equals $\mu - 1$ and OC_0 determines $\epsilon_1 \omega^2/c^2$, where $\mu = \epsilon_2/\epsilon_1$ (see text).

limit; no states exist to the left of OB. For the points between OB and the δ axis, the A material is locally of nonpropagating nature, while B allows local propagation. By comparing Eqs. (1) and (2), we see that $\omega^2\epsilon_1/c^2$ and $\omega^2\epsilon_1(\mu-1)/c^2$ corresponds to $2m(E-E_A)/\hbar^2$ and $2m\delta/\hbar^2$, respectively, where $\mu\equiv\epsilon_2/\epsilon_1$. Thus, every point in the (ω,μ) plane is mapped into a single point in the $(\delta,E-E_A)$ plane and vice versa. In particular, we see, by taking the quotient to eliminate ω^2, that the lines of constant μ (and variable ω^2) are mapped into the lines

$$\delta=(\mu-1)(E-E_A)\,, \tag{3}$$

and the lines of constant ω (and variable μ) are mapped into the lines $E-E_A=$const. For an arbitrary point C in the $(E-E_A,\delta)$ plane, the slope of OC gives $\mu-1$ and its projection to the $E-E_A$ axis OC_0, gives $\hbar^2\omega^2\epsilon_1/2mc^2$ (see Fig. 1). In Fig. 1, we plot schematically the expected behavior of the mobility edge trajectory (MET). For low δ, the MET starts as $E_c-\bar{V}\sim-\delta^2$, where $\bar{V}=xE_B+(1-x)E_A$ and E_c is the mobility edge. For very high δ and for E between E_B and E_A, but not close to them, the behavior of the system is expected to be as in classical particle mechanics (because the wavelength approaches zero). Thus the MET will stay either close to the δ axis, if x is less than x_c, or close to the line OB, if x is larger than x_c, where x_c is critical percolation concentration for the B component. The important question is whether for some range of intermediate values of δ the MET crosses to the positive $E-E_A$ semiplane, as shown in Fig. 1. If this happens, we will have genuine CWL, i.e., one occurring under conditions where both materials allow local propagation (both ϵ_2 and ϵ_1 positive).[12] Unfortunately, all the numerical and analytical work up to now[13] for finding the mobility edge in binary alloy disordered systems has been done in simple lattice systems, where the MET has no possibility to cross the δ axis, unless it reaches the vicinity of an upper band edge.[14]

In the present work, we examine whether classical localization is possible without any help from a real or incipient upper band edge. Thus, we work either in the continuum with no correlations between A and B, or in a more complicated lattice near the lower band edge (and far below the upper band edge), where the differences from the continuum are not so important.

Our model of the random continuum can be pictured as follows: Space is divided into elementary identical polyhedra (e.g., dodecahedra), each one of volume $4\pi a^3/3$, which are filled randomly either by A with probability $1-x$ or by B with probability x. In the calculation of scattering cross sections, the polyhedron is replaced by a sphere of radius a. In the lattice approximating the continuum (which was chosen for simplicity as simple cubic), we employ for the electronic problem a one orbital per site tight-binding Hamiltonian of the form

$$H=\sum_n |n\rangle\epsilon_n\langle n| + \sum_{n,m} t_{nm}|n\rangle\langle m|\,,$$

where t_{nm} is t for n,m nearest neighbors and zero otherwise. In setting up an approximate correspondence between this lattice model and the random continuum, the elementary polyhedra are taken to be cubes of volume L^3. Within each elementary cube, all ϵ_n's are either ϵ_A with probability $1-x$ or ϵ_B with probability x. ($E_{A(B)}$ corresponds to $\epsilon_{A(B)}-6|t|$.) The more lattice points are within each elementary cube, the better our lattice approximates the continuum. We have used $L=2$, 3, and 4 times the lattice spacing corresponding to 8, 27, and 64 lattice points within each elementary cube. To determine the mobility edge trajectory in this 3D tight-binding model, we use the transfer-matrix method.[15,16] This is the most reliable technique in obtaining quantitative results[15,16] in the problem of Anderson localization in disordered systems. In the transfer-matrix method, one considers cylinders of square cross section M^2 made from our random lattice model. For each cylinder of length N, one determines numerically the largest localization length λ_M as $N\to\infty$. At the mobility edge[15,16] $\lambda_M/M=0.6$, while for extended (localized) states λ_M/M vs M increases (decreases), respectively, with increasing M. For our studies here, we have used $M=9,12$ for $L=3$, $M=8,10,12$ for $L=2$, and $M=12$ for $L=4$, and N up to 6000. Most of our data were obtained for $L=3$, $M=9$, $N=6000$. The mobility edge trajectory was determined with the criterion $\lambda_M/M>0.6$ or <0.6 for extended or localized states, respectively. Our numerical results for the MET are shown in Fig. 2 for $x=0.10$ (solid line). The MET crosses to the positive $E-E_A$ semiplane twice showing that classical wave localization occurs in our lattice model although under rather severe conditions. Indeed, it follows from the data in Fig. 2 and Eq. (3) that in order to have classical wave localization $\mu\equiv\epsilon_2/\epsilon_1$ must exceed 14 ± 2 and the dimensionless fre-

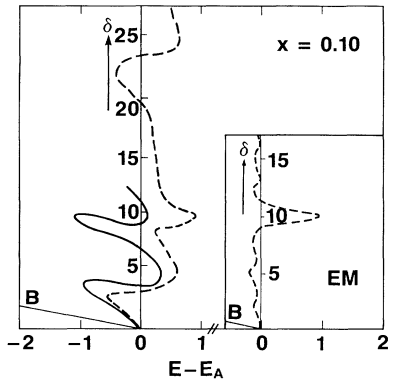

FIG. 2. Mobility edge trajectory obtained from numerical results (solid line) and CPA results (dashed line) for the scalar case with $x=0.10$. Inset: The CPA mobility edge trajectory for the electromagnetic wave case for $x=0.1$. The units of δ and $E-E_A$ are equal to $\hbar^2/2ma^2$. The accuracy of the correspondence of our tight-binding model with the continuum model breaks down for $\delta>10$ (solid line).

quency $\sqrt{\epsilon_1}\omega a/c$ must be around the value $2.2/(\mu-1)^{1/2}<0.65$. According to our data in Fig. 2, classical wave localization will also appear between the upper crossings of the MET which correspond to $\sqrt{\epsilon_1}\omega a/c$ being around the value $3.2/(\mu-1)^{1/2}$; however, this higher localized band will appear only when $\mu>45$. To make sure that the crossings shown in Fig. 2 are not artifacts associated with the discrete nature of our lattice model, we have done calculations not only for $L=3$, but for $L=4$. The rich structure shown in Fig. 2 remains for the $L=4$ case, although the first localized frequency band is moved up a little and centered around $2.4/(\mu-1)^{1/2}$. The minimum value of μ is again 14 ± 2. We have also checked our $M=9$ results by repeating the calculations for $M=12$; no significant variations were found.

The crossings of the MET to the right of the $E=E_A$ line are associated with resonances in the scattering cross section by a B cube of volume L^3 embedded in the average material. We have verified this by correlating the values of δ for which a new bound level appears to the values where the MET exhibits its pronounced structure.

We have tried to determine the optimum value of x for classical wave localization. We have found that for $x=25\%$, there is no crossing of the MET with the δ axis, and therefore no classical localization. It seems that the optimum value of x is about 0.13 ± 0.03, which means that the average thickness of the A material between the B cubes equals their size.

It must be pointed out that our lattice model does not possess sharp boundaries between A and B regions (because of the finite lattice spacing). As a result, localization can be expected to set in for our model at higher values of μ than for the continuum. Moreover, at smaller x the B clusters have in our model a range of shapes which smears out the resonance, again requiring higher dielectric contrast for localization than in the continuum. Finally, at a larger x, the B channel is smoother in our model than in the continuum, once again requiring higher μ for localization. These two latter effects lower the optimum value of x in our model below that for spheres in the continuum. For these reasons, we conclude that our numerical results provide for the first time reliable evidence that genuine classical localization does indeed occur.

In addition to our reliable numerical approach, we have studied the problem by employing the potential-well analogy (PWA)[17] together with a simple coherent potential approximation (CPA).[18] The latter replaces the composite random system by an *homogeneous* effective medium characterized by an energy-dependent complex self-energy $\Sigma(E)$. We determine $\Sigma(E)$ either by demanding that $\langle f(0)\rangle=0$, where $\langle f(0)\rangle$ is the average forward-scattering amplitude,[19] or from the condition that the total cross section is zero.[20] The scattering is due to the replacement of the effective medium within a polyhedron by either A or B. To solve the equation $\langle f(0)\rangle=0$ we transform it to an iterative equation of the form $q_{n+1}=q_n+A\langle f(0;q_n)\rangle$, where $q_n=(E-\Sigma_n)2m/\hbar^2]^{1/2}$, n is the order of the iteration, and A is chosen using the weak scattering limit and demanding as good a convergence as possible. We have used $A=-3/8\pi q_n$. After a successful convergence of q, which implies $\langle f(0;q)\rangle=0$, the mean free path $l=0.5/\mathrm{Im}(q)$ and the renormalized wave vector $k=\mathrm{Re}(q)$ are determined. In the PWA formalism,[17] there is a mobility edge where $kl\simeq0.84$. Using the above method, we calculated the MET for the scalar as well as the vector electromagnetic (EM) wave equation in 3D. We have found that the CPA based on equating the total cross section to zero converges faster than the one that puts the forward-scattering amplitude equal to zero. Both conditions give very similar results for q when x is not very large or when disorder is weak. In Fig. 2, we plot the MET based on PWA and our simple CPA for $x=0.10$ (the dashed line is for scalar case; the EM is shown in the inset). Notice that in the EM case, for $x=0.10$, the only pronounced structure is at $\delta\simeq10$. We have verified that this structure, which is also present in the scalar case (dashed line), is due to the $l=1$ component.[19] The lower structure seen only in the scalar case at $\delta=5$ is due to the $l=0$ component. These CPA-based values of δ coincide with those of the numerical results. In addition to these low-lying localized regions, CWL may appear for higher values of δ corresponding to the second and the third resonances of the $l=0$ (for the scalar case only) and $l=1$ components as well as the first three resonances of the $l=2$ component. The value of x required for the appearance of CWL and the optimum value of x both increase with the order of the resonance. Our CPA for the scalar case gives $x_{\mathrm{opt}}\simeq0.20$, 0.30, and 0.35 with corresponding minimum values of $\mu\simeq6.7$, 6.2, and 9 for the first three resonances. For the EM case, we find $\mu_{\mathrm{min}}\simeq2.8$ at $x_{\mathrm{opt}}\simeq0.6$ for $\omega a\sqrt{\epsilon_1}/c\simeq4.1/(\mu-1)^{1/2}$. For these values of x and μ, the $l=1$ and the $l=2$ resonances have merged together. However, because we are limited to computations with $L<4$, we are unable to check the above CPA results by our numerical methods except for the lowest resonance ($l=0$), where the CPA gives easier localization and higher x_{opt}. More explicitly, the numerical work shows no localization at the lowest resonance for $x>0.23$, while the CPA shows localization even at $x=0.40$.

In conclusion, we have demonstrated for the first time, using reliable numerical techniques, the classical wave localization in composite random binary systems does exist. The one to one correspondence between the electronic and the classical wave problems as summarized in Fig. 1 is very useful in expanding our understanding of the classical wave localization and in allowing reasonable semiquantitative predictions of the suitable range of parameters for its occurrence. Finally, our simple CPA seems to work at least semiquantitatively and is easily

generalizable to the vector electromagnetic wave.

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[12]It should be mentioned that it is easier to have localization in the presence of negative ϵ_1: For $x < x_c$ this would be the trivial case of the wave in a cavity of ϵ_2 material surrounded by the nonpropagating ϵ_1 material; for $x > x_c$ we will have a case like the one studied in Ref. 10, where, as can be seen from Fig. 1, we would always have classical wave localization as $\mu \rightarrow 0^-$.

[13]For correlated disorder in 1D, see E. N. Economou, C. M. Soukoulis, and M. H. Cohen, Phys. Rev. B **37**, 4399 (1988).

[14]The existence of an upper band edge in the ordered spectrum forces the MET to turn around and meet the $E-E_A$ axis again at the upper band edge. As a result, there will always be a crossing with the line OC; hence, there will be CWL, if the ordered spectrum has a gap. On the other hand, in a *continuum* and for $\delta=0$ there is no gap in the spectrum and the mobility edge trajectory will never touch the $E-E_A$ axis again. One may create a gap at some finite δ by constructing a periodic $A_{1-x}B_x$ configuration. If this gap could be made to appear at relatively low δ and relatively large $E-E_A$, it would force the mobility edge trajectory (resulting from a weak disordering process) to bend around towards the $E-E_A$ axis, thus facilitating the classical wave localization. This possibility has been examined recently by S. John, Phys. Rev. Lett. **58**, 2489 (1987); S. John and R. Rangavajan, Phys. Rev. B **38**, 10101 (1988).

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[19]For the scalar wave, the forward-scattering amplitude from a sphere is given by $f(0)=i\sum_l(2l+1)r_l/q$, while for the vector electromagnetic (EM) wave $f(0)=i\sum_{l=1}(2l+1)/(r_l+\bar{r}_l)/2q$, where

$$r_l=\frac{qj_l(ka)j_l'(qa)-kj_l(qa)j_l'(ka)}{qj_l(ka)h_l'(qa)-kh_l(qa)j_l'(ka)},$$

$$\bar{r}_l=\frac{kj_l(ka)j_l'(qa)-qj_l(qa)j_l'(ka)}{kj_l(ka)h_l'(qa)-qh_l(qa)j_l'(ka)},$$

j_l and h_l are the spherical Bessel functions of complex arguments, and the prime denotes differentiation with respect to the argument of the Bessel function. Note that the summation in the EM case starts with $l=1$.

[20]When the propagation constant $[2m(E-\Sigma)/\hbar^2]^{1/2}$ outside the scattering sphere is complex, the forward-scattering amplitude is not directly related to the total cross section. The latter could be defined as the total normalized flux of the outgoing spherical wave just outside the sphere plus the normalized "absorption" (if any) within the sphere minus the normalized "absorption" of the incident wave within a sphere of equal size and propagation constant $[2m(E-\Sigma)/\hbar^2]^{1/2}$.