Conformal Gravity in Three Dimensions as a Gauge Theory

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We show that conformally invariant gravity in three dimensions is equivalent to the Yang-Mills gauge theory of the conformal group in three dimensions, with a Chern-Simons action. This means that conformal gravity is finite and exactly soluble.

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General relativity in three dimensions is equivalent to a gauge theory, with a pure Chern-Simons action, and a gauge group ISO(2,1), SO(3,1), or SO(2,2) depending on the value of the cosmological constant. (This observation was anticipated in work on the group manifold approach to supergravity by Achúcarro and Townsend.[1] It was made independently and perhaps more explicitly in Ref. 2, and has been used in very interesting new work by Blencowe on higher-spin particles in three dimensions.[3]) It was conjectured in Ref. 2 that a similar result holds for three-dimensional conformal gravity. We will show in this paper that this is indeed the case.

In this paper, space-time is an oriented three-dimensional manifold M, with either Euclidean $(+++)$ or Lorentzian $(-++)$ signature. Because we do not specify the signature, a number of formulas will contain factors of $(-1)^s$, where s is the number of negative eigenvalues of the metric. In most discussions, however, we assume for brevity that M has Lorentzian signature. The corresponding comments about Euclidean signature follow easily.

A conformally invariant version of conventional gravity on the manifold M can be found using a type of Chern-Simons action.[4,5] In that formulation, the basic variable in the theory is the vielbein $e_i{}^a$. (Indices i,j,k will be "world" indices and a,b,c will be "Lorentz" indices.) The spin connection $\omega_i{}^a$ is defined as a function of $e_i{}^a$ by requiring that

$$D_i e_j{}^a - D_j e_i{}^a = 0 . \tag{1}$$

The topological Chern-Simons action can be constructed from the Riemann curvature tensor. The result is the action

$$I = \int_M \epsilon^{ijk} [\omega_{ia} (\partial_j \omega_k{}^a - \partial_k \omega_j{}^a) + \tfrac{2}{3} \epsilon^{abc} \omega_{ia} \omega_{jb} \omega_{kc}] . \tag{2}$$

This action closely resembles the Chern-Simons action for a Yang-Mills theory. However, the spin connection is not an independent variable, but is instead a function of $e_i{}^a$ defined implicitly in (1). The supersymmetric extension of this action has been described in Refs. 6 and 7, which treat the theory as a gauge theory. However, (1) is still imposed as an external constraint.

In four or more dimensions, space-time is conformally flat if and only if the Weyl tensor vanishes. In three dimensions, the Weyl tensor vanishes identically, and instead space-time is flat if and only if[8]

$$D_k W_{ij} - D_j W_{ik} = 0 , \tag{3}$$

where $W_{ij} = R_{ij} - \frac{1}{4} g_{ij} R$. If we vary (2) with respect to $e_i{}^a$ to find the equation of motion, we discover that the equation of motion is precisely (3). In other words, the equation of motion forces space-time to be conformally flat.

The conformal group is defined as the group of diffeomorphisms of compactified Minkowski space that leave the metric invariant up to a Weyl rescaling, $g_{ij} \to \Lambda(x) g_{ij}$. In three dimensions, the conformal group has ten generators and is isomorphic to SO(3,2) in the case of Lorentzian signature [or SO(4,1) for Euclidean signature]. The Poincaré group ISO(2,1) [or ISO(3)], is a subgroup of the conformal group. Thus, the generators of the conformal group include the generators P_a of translation, and the generators J_{ab} of Lorentz transformations. Three new generators K_a generate special conformal transformations, and one, D, generates dilations. The commutation relations for the generators are

$$\begin{aligned}
&[P_a, P_b] = [J_{ab}, D] = [K_a, K_b] = 0 , \\
&[P_a, J_{bc}] = \eta_{ac} P_b - \eta_{ab} P_c , \\
&[K_a, J_{bc}] = \eta_{ac} K_b - \eta_{ab} K_c , \\
&[P_a, K_b] = J_{ab} + \eta_{ab} D , \\
&[P_a, D] = P_a , \quad [K_a, D] = -K_a , \\
&[J_{ab}, J_{cd}] = \eta_{ac} J_{bc} - \eta_{ad} J_{bc} + \eta_{bd} J_{ac} - \eta_{bc} J_{ad} .
\end{aligned} \tag{4}$$

On an oriented three-dimensional manifold, the naturally defined volume fom ϵ^{abc} allows us to redefine the rotation generators $J^a = \frac{1}{2} \epsilon^{abc} J_{bc}$. The inverse of this equation is $J_{ab} = (-1)^s \epsilon_{abc} J^c$. The commutation relations with the new J are

$$[P_a,P_b]=0=[J_a,D]=[K_a,K_b]\,,$$
$$[P_a,J_b]=\epsilon_{abc}P^c,\quad [K_a,J_b]=\epsilon_{abc}K^c\,,$$
$$[P_a,K_b]=(-1)^s\epsilon_{abc}J^c+\eta_{ab}D\,, \tag{5}$$
$$[P_a,D]=P_a,\quad [K_a,D]=-K_a\,,$$
$$[J_a,J_b]=\epsilon_{abc}J^c\,.$$

The isomorphism between the conformal group and SO(3,2) is more obvious if we define new generators, $X_a=2^{-1/2}(P_a+K_a)$, and $Y_a=2^{-1/2}(P_a-K_a)$. The conformal generators can then be neatly arranged as a 5×5 matrix,

$$R_{ab}=\begin{pmatrix} 0 & (-1)^sJ_2 & -(-1)^sJ_1 & -X_0 & Y_0 \\ -(-1)^sJ_2 & 0 & J_0 & -X_1 & Y_1 \\ (-1)^sJ_1 & -J_0 & 0 & -X_2 & Y_2 \\ X_0 & X_1 & X_2 & 0 & D \\ -Y_0 & -Y_1 & -Y_3 & -D & 0 \end{pmatrix}. \tag{6}$$

The metric for this representation is $g_{ab}=\mathrm{diag}((-1)^s,+,+,+,-)$.

As we will show, the action that gives conformal gravity is simply the Chern-Simons action for the gauge group SO(3,2). The Chern-Simons action for a Yang-Mills gauge theory is defined to be

$$I_{\rm CS}=\frac{k}{4\pi}\int_M \mathrm{Tr}(A\wedge dA+\tfrac{2}{3}A\wedge A\wedge A)\,, \tag{7}$$

where k is an integer and A is the gauge field. The Chern-Simons action is invariant under gauge transformations that are connected to the identity, but since $\pi_3(\mathrm{SO}(3,2))\simeq Z$, there are gauge transformations with nonzero winding numbers under which it is not invariant. This will, as in Refs. 4 and 9, lead to quantization of the coupling constant of three-dimensional conformal gravity. We will not address the rather subtle issue of the allowed integer values of k for the group SO(3,2).

Since the SO(3,2) Lie algebra is semisimple, it possesses (up to normalization) a unique gauge-invariant bilinear form, and thus there is up to normalization only one Chern-Simons form. (This contrasts with the case of ordinary general relativity in 2+1 dimensions, which possesses an extra coupling constant at the quantum level.[2]) The invariant quadratic form of SO(3,2) is of the form $(T_a,T_b)=\mathrm{Tr}(T_aT_b)$, the trace being taken in an arbitrary matrix representation, such as the representation in (6). This invariant quadratic form corresponds to the invariant bilinear expression

$$W=\tfrac{1}{2}[P_aK^a+K_aP^a+(-1)^sJ_aJ^a-DD] \tag{8}$$

in the group generators.

We can now introduce the gauge theory of SO(3,2). Define the gauge field as a Lie-algebra-valued one-form

$$A_i=e_i{}^aP_a+\omega_i{}^aJ_a+\lambda_i{}^aK_a+\phi_iD\,. \tag{9}$$

The generator of infinitesimal gauge transformations is a Lie-algebra-valued zero-form

$$u=\rho^aP_a+\tau^aJ_a+\sigma^aK_a+\gamma D\,. \tag{10}$$

The variation of the gauge field A_i under a gauge transformation generated by u is

$$\delta A_i=-D_iu\,, \tag{11}$$

where covariant derivatives are defined by $D_iu=\partial_iu+[A_i,u]$. Concretely, (11) means that the component fields transform as

$$\delta e_i{}^a=-\partial_i\rho^a-\epsilon^{abc}(e_{ib}\tau_c+\omega_{ib}\rho_c)-e_i{}^a\gamma+\phi_i\rho^a\,, \tag{12}$$

$$\delta\omega_i{}^a=-\partial_i\tau^a-\epsilon^{abc}\omega_{ib}\tau_c-(-1)^s\epsilon^{abc}(e_{ib}\sigma_c+\lambda_{ib}\rho_c)\,, \tag{13}$$

$$\delta\lambda_i{}^a=-\partial_i\sigma^a-\epsilon^{abc}(\lambda_{ib}\tau_c+\omega_{ib}\sigma_c)+\lambda_i{}^a\gamma-\phi_i\sigma^a\,, \tag{14}$$

$$\delta\phi_i=-\partial_i\gamma-e_i{}^a\sigma_a+\lambda_i{}^a\rho_a\,. \tag{15}$$

The curvature tensor F_{ij} is defined to be

$$\begin{aligned}F_{ij}=[D_i,D_j]&=\partial_iA_j-\partial_jA_i+[A_i,A_j]\\ &=P_a[\partial_ie_j{}^a-\partial_je_i{}^a+\epsilon^{abc}(e_{ib}\omega_{jc}+\omega_{ib}e_{jc})+e_i{}^a\phi_j-\phi_ie_j{}^a]\\ &\quad+J_a[\partial_i\omega_j{}^a-\partial_j\omega_i{}^a+\epsilon^{abc}\omega_{ib}\omega_{jc}+(-1)^s\epsilon^{abc}(e_{ib}\lambda_{jc}+\lambda_{ib}e_{jc})]\\ &\quad+K_a[\partial_i\lambda_j{}^a-\partial_j\lambda_i{}^a+\epsilon^{abc}(\lambda_{ib}\omega_{jc}+\omega_{ib}\lambda_{jc})-\lambda_i{}^a\phi_j+\phi_i\lambda_j{}^a]\;+D(\partial_i\phi_j-\partial_j\phi_i+e_i{}^a\lambda_{ja}-\lambda_i{}^ae_{ja})\,.\end{aligned} \tag{16}$$

We can now construct the Chern-Simons action

$$\begin{aligned}I&=\frac{k}{8\pi}\int_M\epsilon^{ijk}\mathrm{Tr}\{A_i(\partial_jA_k-\partial_kA_j)+\tfrac{2}{3}A_i[A_j,A_k]\}\\ &=\frac{k}{8\pi}\int_M\epsilon^{ijk}[e_{ia}(\partial_j\lambda_k{}^a-\partial_k\lambda_j{}^a)+\tfrac{1}{2}(-1)^s\omega_{ia}(\partial_j\omega_k{}^a-\partial_k\omega_j{}^a)-\phi_i(\partial_j\phi_k-\partial_k\phi_j)\\ &\qquad+2\epsilon^{abc}e_{ia}\omega_{jb}\lambda_{kc}+2\phi_i\lambda_j{}^ae_{ka}+\tfrac{1}{3}(-1)^s\epsilon^{abc}\omega_{ia}\omega_{jb}\omega_{kc}]\,.\end{aligned} \tag{17}$$

The equations of motion derived from this action are $F_{ij}=0$, or

$$\partial_i e_j{}^a - \partial_j e_i{}^a + \epsilon^{abc}(e_{ib}\omega_{jc}+\omega_{ib}e_{jc}) + e_i{}^a\phi_j - \phi_i e_j{}^a = 0\,, \tag{18}$$

$$\partial_i \omega_j{}^a - \partial_j \omega_i{}^a + \epsilon^{abc}\omega_{ib}\omega_{jc} + (-1)^s\epsilon^{abc}(e_{ib}\lambda_{jc}+\lambda_{ib}e_{jc}) = 0\,, \tag{19}$$

$$\partial_i \lambda_j{}^a - \partial_j \lambda_i{}^a + \epsilon^{abc}(\lambda_{ib}\omega_{jc}+\omega_{ib}\lambda_{jc}) - \lambda_i{}^a\phi_j + \phi_i\lambda_j{}^a = 0\,, \tag{20}$$

$$\partial_i\phi_j - \partial_j\phi_i + e_i{}^a\lambda_{ja} - \lambda_i{}^a e_{ja} = 0\,. \tag{21}$$

Let us explore the gauge transformations (12)-(15). If we set $\rho^a=\sigma^a=\gamma=0$, the τ^a transformation gives a local Lorentz transformation. Similarly, if we set $\rho^a=\tau^a=\sigma^a=0$, the γ transformation is a local rescaling, or conformal transformation. Thus, local Lorentz transformations and conformal transformations are gauge transformations.

Local diffeomorphisms are not apparent in the transformation laws. A diffeomorphism generated by a vector field $-v^k$ generates the standard transformations

$$\tilde\delta e_i{}^a = -v^k(\partial_k e_i{}^a - \partial_i e_k{}^a) - \partial_i(v^k e_k{}^a)\,,\quad \tilde\delta\omega_i{}^a = -v^k(\partial_k\omega_i{}^a - \partial_i\omega_k{}^a) - \partial_i(v^k\omega_k{}^a)\,,$$
$$\tilde\delta\lambda_i{}^a = -v^k(\partial_k\lambda_i{}^a - \partial_i\lambda_k{}^a) - \partial_i(v^k\lambda_k{}^a)\,,\quad \tilde\delta\phi_i = -v^k(\partial_k\phi_i - \partial_i\phi_k) - \partial_i(v^k\phi_k)\,. \tag{22}$$

This should be a gauge transformation in the theory of conformal gravity. If we make the gauge transformation with gauge parameters $\rho^a=v^k e_k{}^a$, $\tau^a=v^k\omega_k{}^a$, $\sigma^a=v^k\lambda_k{}^a$, and $\gamma=v^k\phi_k$, we see that the gauge transformation differs from the diffeomorphism by

$$\tilde\delta e_i{}^a - \delta e_i{}^a = v^k[\partial_i e_k{}^a - \partial_k e_i{}^a + \epsilon^{abc}(e_{ib}\omega_{kc}+\omega_{ib}e_{kc}) + e_i{}^a\phi_k - \phi_i e_k{}^a]\,,$$
$$\tilde\delta\omega_i{}^a - \delta\omega_i{}^a = v^k[\partial_i\omega_k{}^a - \partial_k\omega_i{}^a + \epsilon^{abc}\omega_{ib}\omega_{kc} + (-1)^s\epsilon^{abc}(e_{ib}\lambda_{kc}+\lambda_{ib}e_{kc})]\,,$$
$$\tilde\delta\lambda_i{}^a - \delta\lambda_i{}^a = v^k[\partial_i\lambda_k{}^a - \partial_k\lambda_i{}^a + \epsilon^{abc}(\lambda_{ib}\omega_{kc}+\omega_{ib}\lambda_{kc}) - \lambda_i{}^a\phi_k + \phi_i\lambda_k{}^a]\,,$$
$$\tilde\delta\phi_i - \delta\phi_i = v^k(\partial_i\phi_k - \partial_k\phi_i + e_i{}^a\lambda_{ka} - \lambda_i{}^a e_{ka})\,. \tag{23}$$

These all vanish precisely when the equations of motion are satisfied. Thus, local diffeomorphisms are gauge transformations, at least on shell.

Having constructed the gauge theory SO(3,2) with a Chern-Simons action we will now show that this theory is equivalent to the conformal gravity theory of Ref. 4 at the classical level, under the following restriction.

In Ref. 4, the basic variable is the vielbein; the spin connection is defined as a functional of the vielbein; and since the inverse of the vielbein appears in the definition of the spin connection, it is necessary to assume that the vielbein is invertible. This is natural in the context of Riemannian geometry. On the other hand, in our action (17) there is absolutely no reason to assume that the vielbein is invertible; such an assumption would be contrary to the general rules of gauge theory. In fact, in our gauge theory formulation, the question of whether the vielbein is invertible in a given space-time region is gauge dependent. For instance, on any space-time manifold M, the field equations (18), etc., are obviously obeyed by $e=\omega=\phi=\lambda=0$. While this seems as far as possible from an invertible vielbein, it locally can be gauge transformed into a form in which the vielbein is invertible. (Whether this is possible globally depends on the topology of space-time. If space-time is S^3, the solution $e=\cdots=0$ can be gauge transformed into a form in which the vielbein is everywhere invertible, but if space-time is a lens space S^3/Z_k, this is not possible.)

Since the vielbein must be invertible in Ref. 4 while we do not require this, the meaning of the claimed classical equivalence between the two theories is simply that solutions of the field equation (3) of Ref. 4 are in correspondence with solutions of our field equations in which the vielbein (in a suitable gauge) is everywhere invertible. Actually, the different between the two theories—the fact that we permit "singularities" corresponding to a noninvertible vielbein—is extremely important at the quantum level, since (as in the discussion in Ref. 2) the quantum theory derived from (17) is renormalizable and even finite, while this could not be expected in the formulation of Ref. 4.

The invertibility of the vielbein enters as follows. When $e_i{}^a$ is invertible, the gauge transformation law (15) for ϕ_i shows that the σ^a gauge invariance is precisely sufficient to set ϕ_i to 0. Furthermore, (12) says that $e_i{}^a$ is completely unchanged by a σ^a gauge transformation, and so $e_i{}^a$ remains invertible in this new gauge.

With the gauge choice $\phi_i=0$, the equations of motion simplify considerably. The equation of motion for the vielbein becomes

$$\partial_i e_j{}^a - \partial_j e_i{}^a + \epsilon^{abc}(e_{ib}\omega_{jc}+\omega_{ib}e_{jc}) = 0\,. \tag{24}$$

This is identical to

$$D_i e_j{}^a - D_j e_i{}^a = 0\,, \tag{25}$$

where D_i is the covariant derivative with respect to the spin connection $\omega_i{}^a$. In the gauge $\phi_i=0$ (but not otherwise), $\omega_i{}^a$ is precisely the Levi-Civita connection. The other equations of motion become

$$D_i\lambda_j{}^a - D_j\lambda_i{}^a = 0\,, \tag{26}$$

$$e_i{}^a\lambda_{ja} - e_j{}^a\lambda_{ia} = 0\,, \tag{27}$$

$$R_{ij}{}^a + (-1)^s\epsilon^{abc}(e_{ib}\lambda_{jc} + \lambda_{ib}e_{jc}) = 0\,, \tag{28}$$

where the Riemann curvature tensor is defined in the standard way as

$$R_{ij}{}^a = \partial_i\omega_j{}^a - \partial_j\omega_i{}^a + \epsilon^{abc}\omega_{ib}\omega_{jc}\,. \tag{29}$$

Until this point in the discussion of the gauge theory of SO(3,2), we have not introduced the Christoffel symbols of general relativity. To compare with the formulation in (3), it is convenient to introduce them. The Christoffel symbols may be defined by requiring that

$$D_i e_j{}^a = 0\,, \tag{30}$$

where the covariant derivative is defined by

$$D_i e_j{}^a = \partial_i e_j{}^a - \Gamma^k{}_{ij}e_k{}^a + \omega_i{}^a{}_b e_j{}^b\,.$$

[The point is that if one does not wish to introduce the Christoffel symbols, (25), in which the Christoffel symbols do not contribute, defines the connection ω; but the stronger condition (30) serves to define both ω and Γ. With the metric defined by $g_{ij}=e_{ia}e_j{}^a$, (30) implies $D_k g_{ij}=0$, an alternative conventional definition of the Christoffel symbols.]

The Ricci tensor is now

$$R_{ij} = g^{kl}R_{ikjl} = g^{kl}e_j{}^a e_l{}^b R_{ikab} = g^{kl}e_j{}^a e_l{}^b\epsilon_{abc}R_{ik}{}^c = -e_j{}^a\lambda_{ia} - g_{ij}e^{ka}\lambda_{ka}\,. \tag{31}$$

In the last step, (28) has been used; the resulting expression for R_{ij} is symmetric by virtue of (27). The conformal tensor W_{ij} that appears in (3) now becomes

$$W_{ij} = -e_j{}^a\lambda_{ia} = -e_i{}^a\lambda_{ja}\,. \tag{32}$$

Thus,

$$D_k W_{ij} - D_j W_{ik} = -D_k e_i{}^a\lambda_{ja} - e_i{}^a D_k\lambda_{ja} + D_j e_i{}^a\lambda_{ka} + e_i{}^a D_j\lambda_{ka} = D_j e_i{}^a\lambda_{ka} - D_k e_i{}^a\lambda_{ja} + e_i{}^a(D_j\lambda_{ka} - D_k\lambda_{ja}) = 0\,, \tag{33}$$

and space-time is conformally flat.

This shows that a solution of the equations derived from (17) in which the vielbein is invertible gives a solution of (3). Conversely, given an invertible vielbein $e_i{}^a$ that defines a curvature obeying (3), one obtains a solution of Eqs. (18)–(21) by taking ω to be the Levi-Civita connection, ϕ to be zero, and using (32) [or equivalently (27) and (28)] to define λ. This establishes the desired results.

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