Diverse Manifolds in Random Media

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We suggest a unifying perspective of various manifolds subject to quenched correlated disorder. The new picture incorporates seemingly disparate problems as domain walls in impurity-stricken Ising magnets and directed walks upon random lattices. Using Flory-style arguments and a functional nonlinear renormalization group, we obtain new results, including the random-bond interfacial roughening exponent $\zeta_{RB}=\frac{2}{9}\epsilon$, where $\epsilon=5-d$, which we believe to be exact, as well as a conjecture for the (2+1)-dimensional directed polymer index $\zeta_{2+1}=\frac{30}{49}$.

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The statistical mechanics of polymers and interfaces in the presence of quenched correlated random disorder is an extraordinarily rich subject, not lacking in controversial theoretical subtleties and frequently begetting healthy, though sometimes bitter debate. A recent manifestation of this problem, due to Kardar and Zhang,[1] concerns the scaling properties of directed polymers in random media. Such polymers are highly anisotropic objects, and are essentially directed walks biased along a single preferred direction with disorder-induced fluctuations constrained entirely to the transverse dimensions. The scaling of these transverse fluctuations is the fundamental issue at hand and raises, rather naturally, very general questions of optimization theory,[2] such as minimal-energy paths. In this regard, it may even shed light on the apparent ultrametric tree structure exhibited by river basins, blood vessels, and neuronal networks.[3] Furthermore, the directed polymer problem is mathematically equivalent to a host of others, among which are interfacial growth dynamics of Eden clusters, large-time behavior of randomly stirred fluids asymptotics of driven diffusion, and the evolution of Sivashinski flame fronts. The basis of this equivalence can be traced to the ubiquity of the Burgers' equation,[4] the simplest generalization of the diffusion equation possessing the requisite nonlinearities. Because of the large payoff, much effort[5,6] has been devoted to the dynamic scaling properties of this equation.

Unfortunately, the directed walk amidst quenched impurities has proved to be quite a difficult problem, indeed! Without disorder, the matter is simple. Transverse fluctuations scale with longitudinal length as $|\mathbf{z}|\sim x^{\zeta}$, with the trivial random-walk index $\zeta_{RW}=\frac{1}{2}$, independent of the number of perpendicular dimensions n. Yet, with pinning impurities present to enhance fluctuations, the roughening exponent is known exactly only for $n=1$, where the polymer behaves as the interface of a 2D Ising model subject to random-bond disorder. For this case, complementary approaches[6-8] have fixed $\zeta_{n=1}=\frac{2}{3}$. Early numerical work by Kardar and Zhang[1] for $n=2,3$ hinted that this value might be superuniversal, as in the pure case. In fact, McKane and Moore[9] have even suggested a mechanism justifying why this might be so. Nevertheless, subsequent, more precise computer work by Wolf and Kertész[10] on Eden growth, when translated into directed polymer language, cast doubt on this notion, yielding $\zeta_{n=2}=0.60\pm0.02$, $\zeta_{n=3}=0.57\pm0.05$, and the conjecture that $\zeta(n)=(n+1)/(2n+1)$. Studies[5,6] of the Burgers' equation reveal, however, that for $n>2$, ζ is either trivial or governed by a nonperturbative fixed point, rendering a smoothly varying wandering exponent an unlikely prospect. Interestingly, though, Derrida and Spohn[11] have recently solved the directed polymer problem on a Cayley tree and their findings indicate that the random-walk value may be retrieved in the infinite-dimension limit.

Somewhat orthogonal to this directed polymer problem is that of domain walls roughened by quenched impurities.[12-14] Here one considers an interface $z=z(\mathbf{x})$ separating two phases of matter in $d=d'+1$ dimensions and studies the fluctuations incurred by random disorder. The associated roughening exponent, defined via $|z|\sim|\mathbf{x}|^{\zeta}$, describes height fluctuations as one scales lengths in the d'-dimensional basal plane. One aspect of this subject, the random-field (RF) Ising model, was recently the subject of controversy. The crux of the matter concerned the lower critical dimension of the system, which was determined by the condition[15] $\zeta_{RF}(d_l)=1$. All parties agreed that above five dimensions, randomness was irrelevant as $\zeta_{RF}(d\geq5)=0$. However, for $\epsilon=5-d>0$, there were two outspoken camps. The first, of the dimensional reduction persuasion,[16] believed $\zeta_{RF}=\epsilon/2$, involving a simple shift from the corresponding pure problem, where thermal fluctuations yield an exponent $\zeta_{TH}=(3-d)/2$. The opposition[17] placed great faith in Imry-Ma domain-wall arguments which gave $\zeta_{RF}=\epsilon/3$. Independent investigations[18] have since conspired to fix $d_l=2$, thereby confirming the Imry-Ma picture. Nonetheless, it remained ill-understood precisely why the Flory-style domain-wall argument worked so well for RF's, since it is nothing but a refined mean-field treatment.

In the midst of all this, Fisher[19] studied the related question of interface fluctuations in the presence of random-bond (RB) disorder and using functional renormalization-group (RG) methods *numerically* estimated the roughening exponent to be $\zeta_{\rm RB} \approx 0.2083\epsilon$. Of course, for $d=2$, the interface and directed polymer problems become one and the same, and Fisher's estimate $\zeta_{n=1} \approx 0.625$ comes quite close to the correct value $\frac{2}{3}$ discussed earlier. Below, we reproduce the latter exactly as the special case of a more general result. Surprisingly, the RB problem appears unamenable to an Imry-Ma approach. In $d=2$, both Kardar[12] and Natterman[13] have addressed this mystery—they consider a model with correlated randomness that interpolates between the RF and RB situations. For sufficiently long-ranged correlations (those falling off no faster than some critical power law), they established the validity of a Flory-determined ζ, while for short-ranged correlations the wandering exponent sticks to its RB value $\frac{2}{3}$.

It is the purpose of this paper to provide a new perspective on these issues by suggesting a unified picture of diverse manifolds in quenched random media. To this end, we introduce the Landau Hamiltonian for an n-component vector field $\mathbf{z}(\mathbf{x})$ with $d'=d-1$ dimensional support, subject to correlated disorder $V(\mathbf{z})$ that pins the field,

$$H=\int d^{d'}x\{(\nu/2)(\nabla \mathbf{z})^2+\sigma V(\mathbf{z})\}.$$

Here ν is the tension parameter discouraging fluctuations and σ gauges the strength of the random potential. For the convenience of readers most familiar with the problem of interfaces subject to quenched randomness, we will continue the custom of using d' and $d-1$ interchangeably, although this is somewhat antithetical to our unified (n,d') picture. Clearly, $d'=1$ corresponds to a directed polymer in $n+1$ dimensions, while $n=1$ yields the interface problem in $d'+1$ dimensions. We consider correlated randomness that is Gaussian with zero mean and variance

$$\langle V(\mathbf{z},\mathbf{x})V(\mathbf{z}',\mathbf{x}')\rangle=\delta^{d'}(\mathbf{x}-\mathbf{x}')R(\mathbf{z}-\mathbf{z}').$$

For the case of uncorrelated impurities (e.g., RB interface problem) the bare $R(\mathbf{z})=\delta^n(\mathbf{z})$, but the fixed-point function $R^*_{\rm SR}(z)$ that dictates the scaling properties is expected to be smoother, though still short ranged (see later). By contrast, the RF Ising model involves, in our formulation, correlations that increase linearly[17,19] at large distances. With functions smooth at small argument and behaving asymptotically as $R(\mathbf{z})\sim|\mathbf{z}|^{-\beta n}$, straightforward dimensional considerations permit us to determine the critical dimension above which the randomness is perturbatively relevant, in the RG sense. Under the rescaling, $\mathbf{x}\to\lambda\mathbf{x}$ and $\mathbf{z}\to\lambda^{\zeta}\mathbf{z}$, it is apparent that the parameters ν and σ have scaling indices $y_\nu=2\zeta+d-3$ and $y_\sigma=\frac{1}{2}(d-1-\beta\zeta n)$, since the assumed variance form dictates $V\to\lambda^{-(d-1+\beta\zeta n)/2}V$. With no randomness, a scale-invariant theory necessitates $y_\nu=0$, so that $\zeta=\zeta_0=(3-d)/2$, the free (or thermal) value alluded to above. Disorder is perturbatively relevant if $y_\sigma>0$ at this fixed point; that is, for $d>d_c(\beta,n)=(2+3\beta n)/(2+\beta n)$. Note that for the interface problem $(n=1)$, we retrieve the accepted values[12] $d_c^{\rm RB}(1,1)=\frac{5}{3}$ and $d_c^{\rm RF}(-1,1)=-1$, while for directed polymers ($d=2,\beta=1$), we find $n_{c2}=2$, as anticipated by work[5,6] on the Burgers' equation. It is our belief that for the physical systems of interest, the wandering exponent does not typically assume its free value below d_c, but rather is controlled by a strong disorder nonperturbative fixed point. This has been suspected[1] in the directed polymer problem, but now in the unified scheme, also appears probable for interfaces. As a first step, Flory arguments, which require both terms of H to scale in the same fashion, allow us to fix $\zeta_F(\beta,n)=-(5-d)/(4+\beta n)$ in the regime of perturbative relevance $(d>d_c)$. At and above $d=5$, $\zeta=0$ and the manifold is not roughened by the impurities. Though this Flory prediction for the RB interface problem $\zeta_F(1,1)=\epsilon/5$ is incorrect,[12] yielding an erroneous $\frac{3}{5}$ in $d=2$, it does give the Imry-Ma value $\zeta_F(-1,1)=\epsilon/3$ for random fields. We shall see below that, for a given value of n, the Flory theory is only valid for a range of β. In Fig. 1, we gather our results and make manifest the intended scheme for the specific case of RB disorder.

An alternative method for the determination of the wandering exponent when weak randomness is perturbatively relevant is via the functional renormalization group. As pointed out by Brézin and Orland,[20] the functional RG is nothing but a calculation of the effective ac-

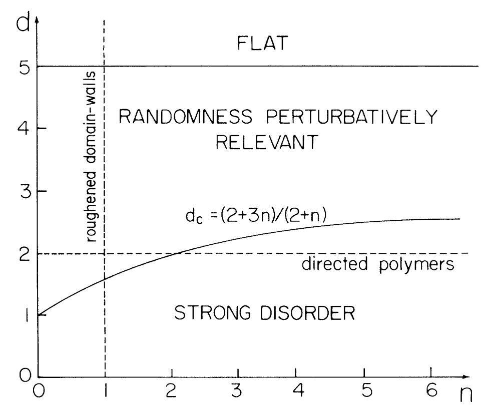

FIG. 1. The effects of random-bond disorder upon manifolds described by an n-component vector field with $d'=d-1$ dimensional support. Two apparently unrelated problems, directed polymers and wandering domain walls, are highlighted. Below five dimensions, disorder is perturbatively relevant and the manifolds are roughened by the pinning impurities. For sufficiently low dimensionality, strong disorder nonperturbative effects can control the fluctuations. Flory arguments determine the boundary between these two regimes.

tion to one-loop order, followed by differential length $\mathbf{x} \rightarrow \mathbf{x}(1+\delta l)$ and field $\mathbf{z} \rightarrow \mathbf{z}(1+\zeta\delta l)$ rescalings, to note how the interaction, in our case $R(\mathbf{z})$, behaves under the infinitesimal dilation. Using replicas to deal with the randomness (averaging over the disorder V introduces R as the potential amongst the replicas), one readily derives the functional differential recursion relation describing the flow of $R(\mathbf{z})$:

$$\frac{\partial R}{\partial l} = (\epsilon - 4\zeta)R + \zeta z R' + \tfrac{1}{2}(R'')^2 - R''R''(0) + \frac{(n-1)}{2}\frac{R'^2}{z} - \frac{(n-1)}{2}\frac{R'}{z}R''(0) + \cdots .$$

Here, primes denote derivatives with respect to z. The first term follows from the dimension of R, the second from the dimension of $\mathbf{z}$, and the remainder is due to expansion of the one-loop logarithm. The ellipsis denotes discarded higher powers in the Taylor series. We are at liberty to fix the scale of R, so we set $R''(0) = -\epsilon$ and search for the invariant fixed point functions (FPF's) which satisfy $\partial_l R^*(z) = 0$. Fisher, who studied the above partial differential equation for $n=1$, noted that the Imry-Ma exponent $\zeta_{\rm RF} = \epsilon/3$ follows from the RF FPF that grows linearly at large z. This is nothing but a specific example of the linearly truncated, essentially dimensional RG that gives rise to the Flory exponents $\zeta_F = \epsilon/(4+\beta n)$ for FPF's behaving asymptotically as $R^* \sim z^{-\beta n}$. We next make a search for the FPF associated with short-range (SR) correlations. With a Gaussian in mind, it is easy to show that the SR FPF falls off as $R^*_{\rm SR}(z) \sim \epsilon z^{-4-n+\epsilon/\zeta} \exp(-\zeta z^2/2\epsilon)$. This damped power law is the basis for many of the results that follow.

Consider first the interface problem ($n=1$). We make the suggestion, entirely in keeping with the basic tenets of critical phenomenon, that provided the correlations in randomness are sufficiently long ranged (i.e., small enough β), the mean-field Flory theory is correct and the roughening exponent is $\zeta_F = \epsilon/(4+\beta)$, as described above. Nevertheless, as the correlations become progressively short ranged, one eventually reaches a critical β_c such that the scaling is controlled entirely by the FPF $R^*_{\rm SR}(z)$, representing a *Gaussian damping of the critical algebraic decay.* For $\beta > \beta_c$, the index ζ sticks to the value characteristic of this SR FPF (the crossover exponent is negative) and abandons the Flory formula. Determination of this critical value is easy since it is only at β_c that ζ simultaneously satisfies the constraints of both Flory and nonlinear functional RG treatments. That is, a glance at $R^*_{\rm SR}$ implies $\beta_c = 5 - \epsilon/\zeta_{\rm SR}$. Substitution of this equation into the Flory formula reveals $\zeta_{\rm SR} = \frac{2}{9}\epsilon$ and $\beta_c = \frac{1}{2}$. Joining this new result to the Flory theory we know to be correct for $\beta \le \beta_c$, we summarize our findings[21] in Fig. 2(a). The interesting new prediction is that $\zeta_{\rm RB} = \frac{2}{9}\epsilon$, compared to the Imry-Ma determined $\zeta_{\rm RF} = \epsilon/3$. As suspected, interfaces are roughened less by RB's than they are by RF's. Correlated disorder incurs greater wandering.

We stress that, though based on the same differential equation, the present methodology differs greatly from that of Fisher, whose procedure involves *numerical* integration of the partial differential equation for $R^*_{\rm SR}(z)$ from the origin (where neglect of the higher-order terms in the ellipsis is not justified) to large z with $\zeta_{\rm SR}$ the sensitive eigenvalue insuring an asymptotically vanishing FPF. By contrast, our use of the partial differential equation is limited strictly to its domain of validity. We seek only the functional form of the tail, so the truncation is legitimate. Indeed, we believe $\zeta_{\rm RB} = \frac{2}{9}\epsilon$ to be exact as higher-loop contributions to the effective action should leave the tail unaltered. Lastly, our assumptions concerning the existence of a β_c, continuity of ζ, and the vestigial link of $R^*_{\rm SR}$ to the critical power law, appear quite natural.

The above scheme is corroborated by the fact that for $\epsilon=3$, we retrieve the exact results of Kardar[12] and Natterman,[13] $\zeta_{n=1} = \frac{2}{3}$ and $\beta_c = \frac{1}{2}$, for the directed polymer problem in 1+1 dimensions. What about directed

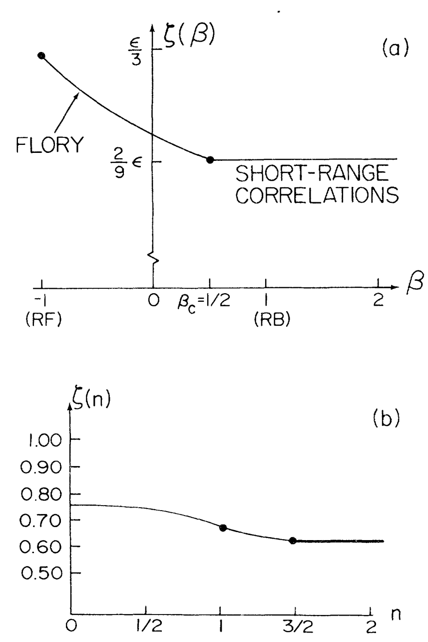

FIG. 2. (a) Wandering exponent ζ for interface roughening as a function of β, which describes the falloff of impurity correlations. Flory theory is correct only for sufficiently long-ranged correlations. Beyond the critical value $\beta_c = \frac{1}{2}$, a single SR fixed point function dictates the scaling behavior and ζ sticks to the RB value $2\epsilon/9$. (b) Directed polymer index in $n+1$ dimensions. The result $\zeta_{n=1} = \frac{2}{3}$ is exact. Furthermore, a unique FPF determines ζ for the entire interval between $n_{c1} = \frac{3}{2}$ and $n_{c2} = 2$.

polymers ($d'=1$) with $n\neq 1$? As mentioned earlier, any anomalous scaling ($\zeta\neq\frac{1}{2}$) for $n>2$ is strictly nonperturbative in nature and therefore beyond the reach of our functional RG. Hence, we restrict ourselves to $n\leq 2$, envisioning an expansion about the well-understood case $n=1$. In addition, we note that analyses[5,6] of the Burgers' equation show a unique FPF controlling the scaling in the range $\frac{3}{2}\leq n\leq 2$, the character of this function revealed as $n\to\frac{3}{2}$. For this reason, we conjecture that throughout this interval, the wandering exponent will stick to its value at $n_{c1}=\frac{3}{2}$. From the nonlinear RG, we know that the SR FPF is a Gaussian damped power law with $\beta_c=4+n-3/\zeta_{\rm SR}$. Of course, at β_c, the Flory formula $\zeta_F=3/(4+\beta n)$ is also satisfied. Simultaneous solution yields $\zeta_{\rm SR}=3(n+1)/(n+2)^2$ and $\beta_c=n/(n+1)$. Note that $\beta_c<1$ for arbitrary n. Hence, the rather cavalier manner in which we characterized δ-function correlated disorder via naive dimensional concerns ($\beta=1$ for RB) was done with complete impunity, since the nonlinear RG maps all bare functions falling faster than β_c onto the same SR FPF—a beautiful manifestation of the renormalization group. Figure 2(b) documents the gentle decline of the directed polymer index $\zeta(n)=\zeta_{\rm SR}$ with increasing n, as well as the break in slope that we anticipate at $n_{c1}=\frac{3}{2}$. These considerations, though quite humble, represent the sole analytic work extent for finite $n\neq 1$ and lead us to the prediction that $\zeta_{2+1}=\frac{30}{49}\approx 0.61$, a value entirely consistent with the recent Monte Carlo simulations[10] of Eden clusters. It would be of much interest to call upon additional nonperturbative tools, such as real-space renormalization or instantons, perhaps, to provide a further test of this result and to gain greater insight into the many-dimensional directed polymer. We are presently investigating these possibilities.

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[1]M. Kardar and Y.-C. Zhang, Phys. Rev. Lett. **58**, 2087 (1987).

[2]Y.-C. Zhang, Phys. Rev. Lett. **59**, 2125 (1987).

[3]B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, New York, 1983).

[4]J. M. Burgers, *The Nonlinear Diffusion Equation* (Reidel, Boston, 1974).

[5]E. Medina, T. Hwa, M. Kardar, and Y.-C. Zhang [MIT Report, 1988 (to be published)] provide an exhaustive RG analysis of the Burgers' equation with correlated noise.

[6]D. Forster, D. R. Nelson, and M. J. Stephen, Phys. Rev. A **16**, 732 (1977).

[7]D. A. Huse, C. L. Henley, and D. Fisher, Phys. Rev. Lett. **55**, 2924 (1985).

[8]M. Kardar, Phys. Rev. Lett. **55**, 2235, 2923 (1985).

[9]A. J. McKane and M. A. Moore, Phys. Rev. Lett. **60**, 527 (1988).

[10]D. E. Wolf and J. Kertész, Europhys. Lett. **4**, 651 (1987).

[11]B. Derrida and H. Spohn, J. Stat. Phys. **51**, 817 (1988).

[12]M. Kardar, J. Appl. Phys. **61**, 3601 (1987).

[13]T. Natterman, Europhys. Lett. **4**, 1241 (1987).

[14]For a review of interface wandering in pure and impure media, see M. E. Fisher, J. Chem. Soc., Faraday Trans. 2 **82**, 1569 (1986).

[15]In this case, no long-range order is possible since it is disrupted by domain walls that meander throughout the system.

[16]D. Mukamel and E. Pytte, Phys. Rev. B **25**, 4779 (1982), and references therein.

[17]G. Grinstein and S.-k. Ma, Phys. Rev. B **28**, 2588 (1983) discuss all aspects.

[18]J. F. Fernandez, G. Grinstein, Y. Imry, and S. Kirkpatrick, Phys. Rev. Lett. **51**, 203 (1983); J. Z. Imbrie, Phys. Rev. Lett. **53**, 1747 (1984); Y.-C. Zhang, J. Phys. A **19**, L941 (1986).

[19]D. Fisher, Phys. Rev. Lett. **56**, 1964 (1986).

[20]E. Brézin and H. Orland, unpublished.

[21]Remember, for β very negative or ϵ large enough, ζ exceeds unity, so our implicitly single-valued $z(\mathbf{x})$ may be suspect at large length scales.